Neural Networks

Langauge and Computation I, Fall 2025

Zhenghao Herbert Zhou, Oct 9th, 2025

Recap from Classification

Logistic Regression (from lecture on Sep 18th)

$$z = \sum_{i=1}^{n} w_i x_i + b \qquad \text{Sigmoid: } \sigma(z) = \frac{1}{1 + e^{-z}} \qquad \text{decision}(\mathbf{x}) = \begin{cases} 1, & \text{if } \sigma(z) > 0.5 \\ 0, & \text{otherwise} \end{cases}$$

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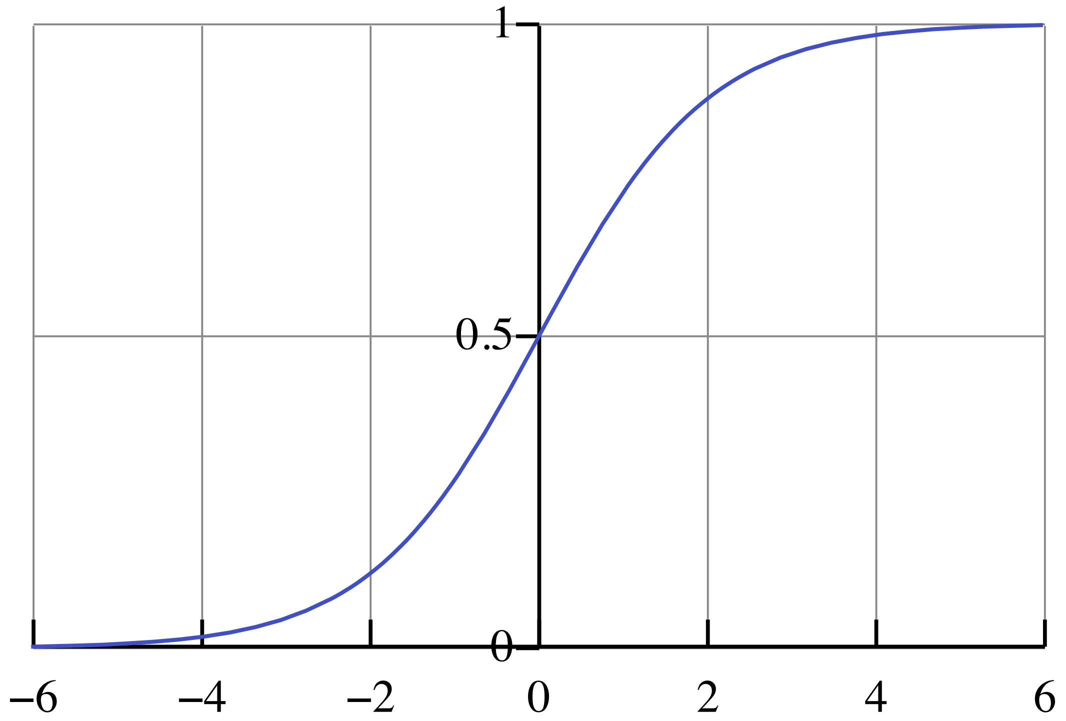

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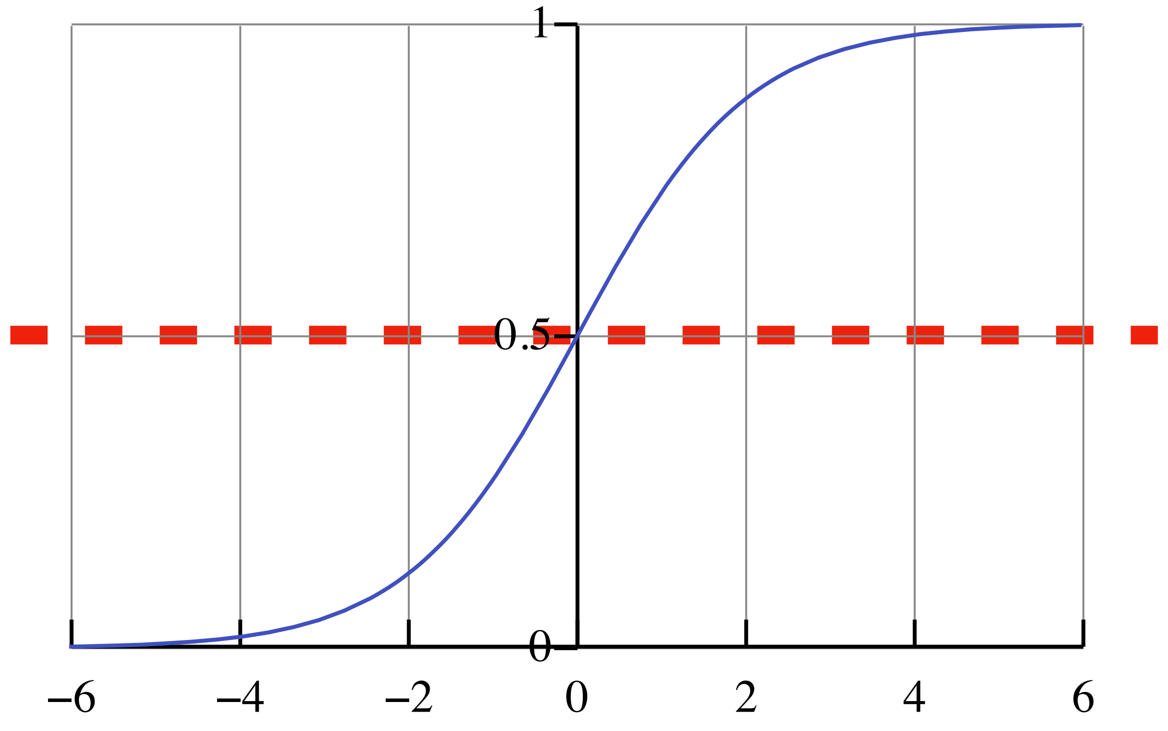

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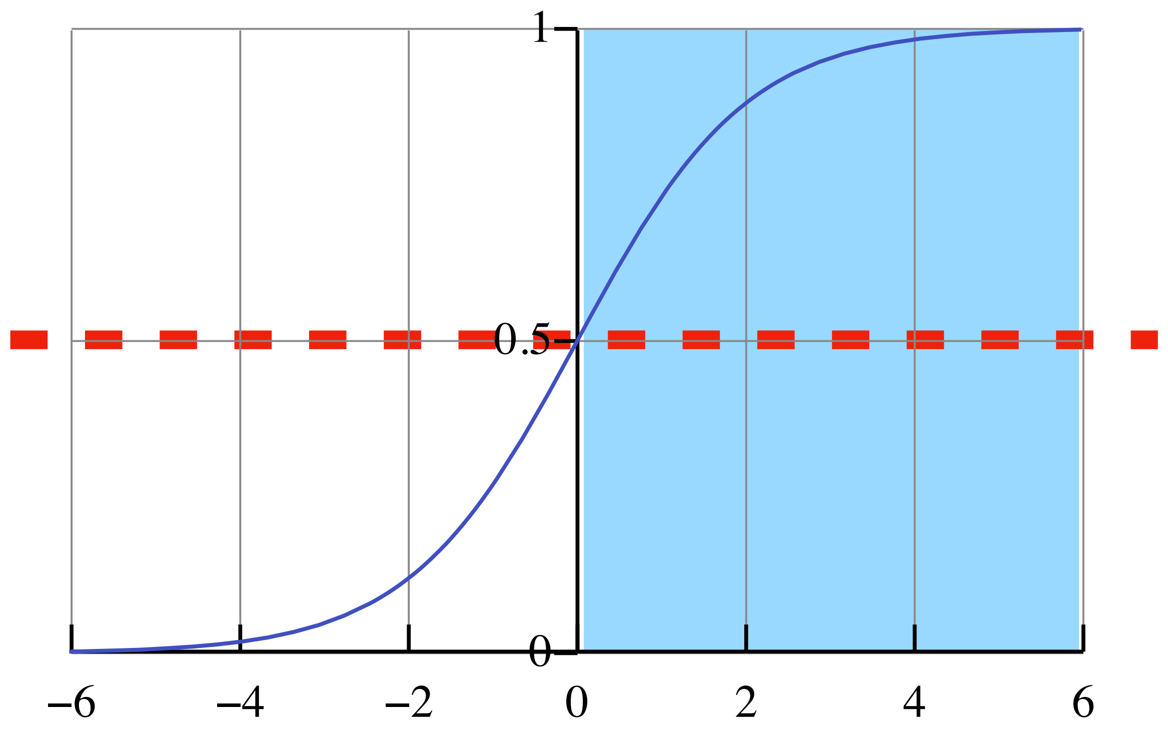

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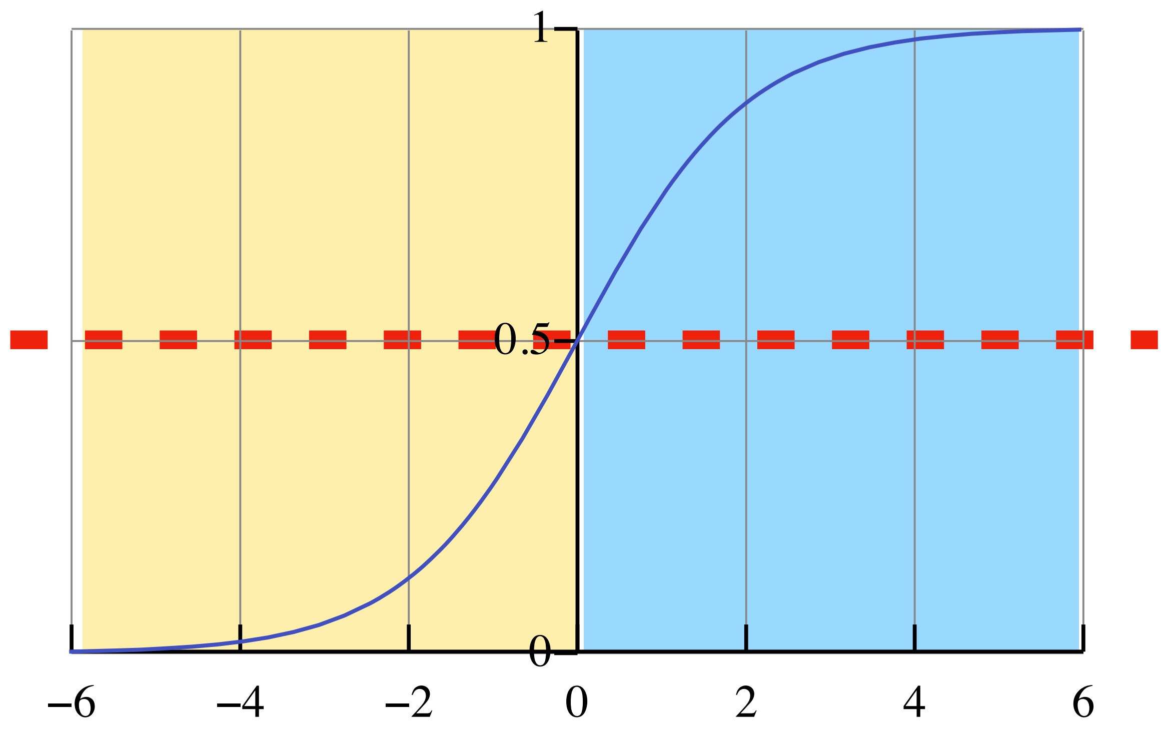

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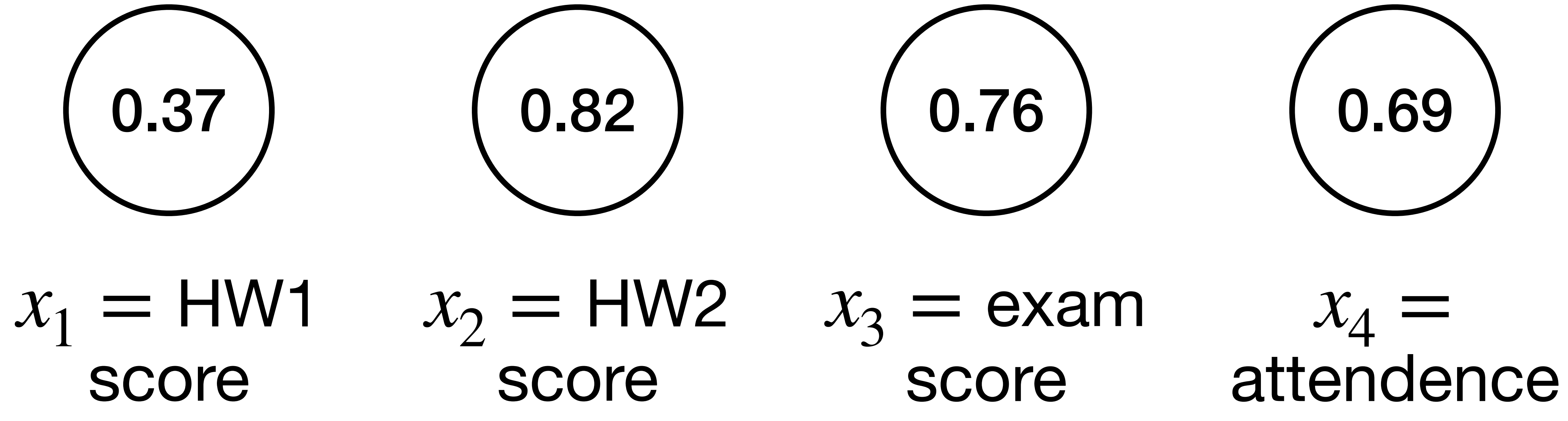

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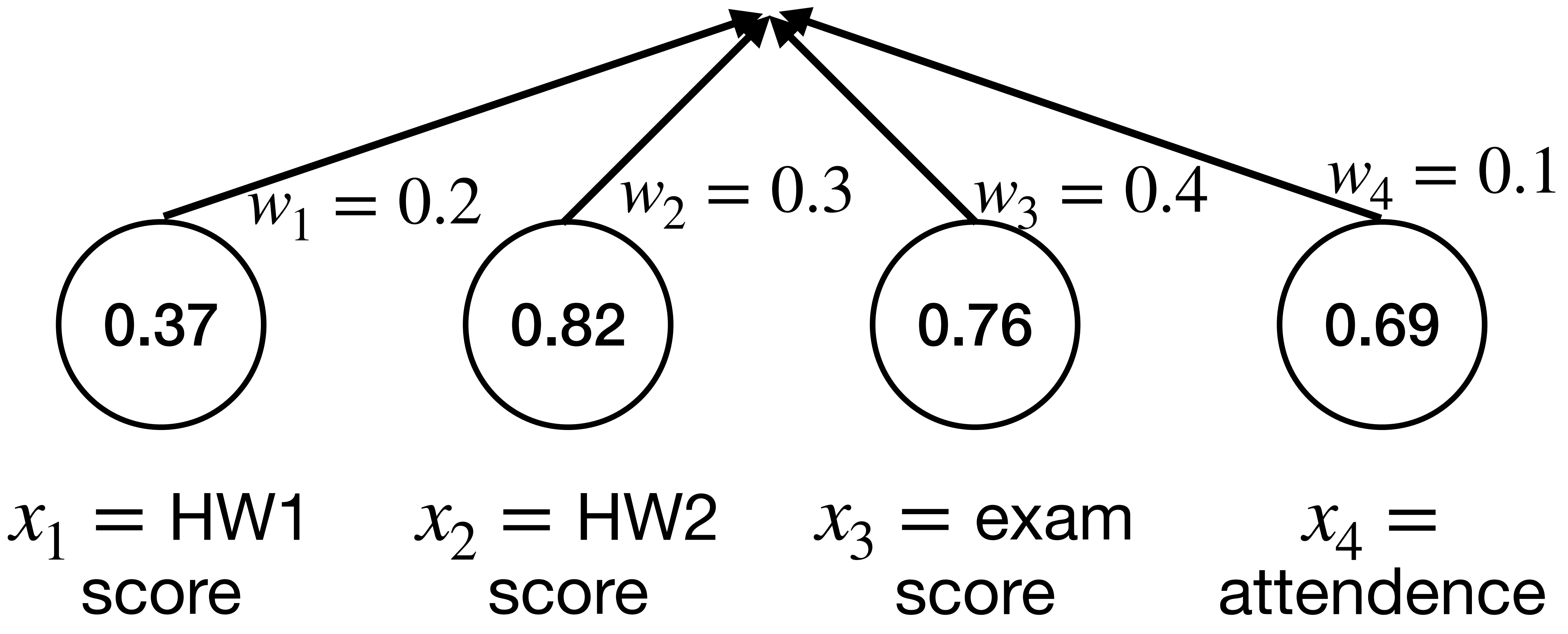

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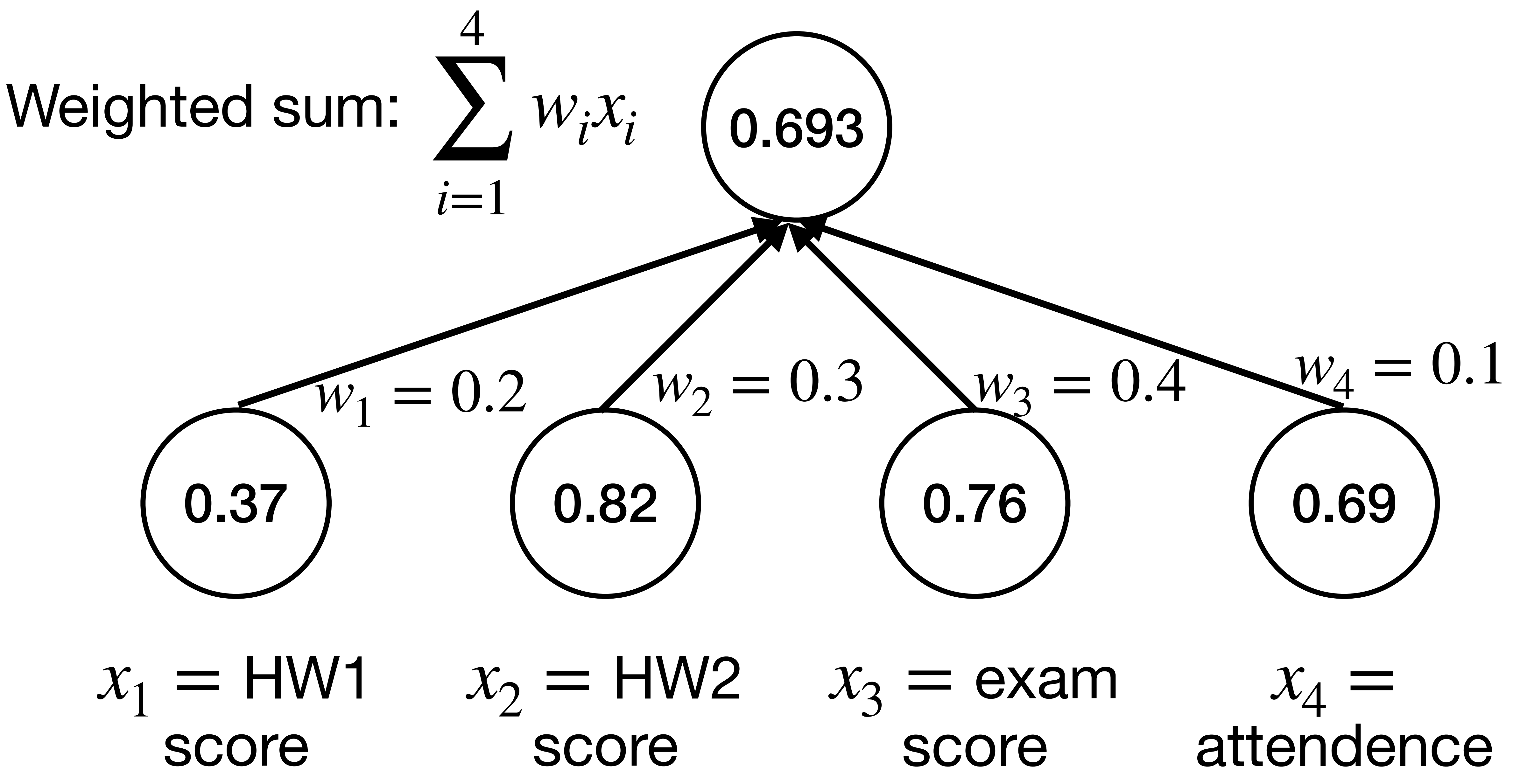

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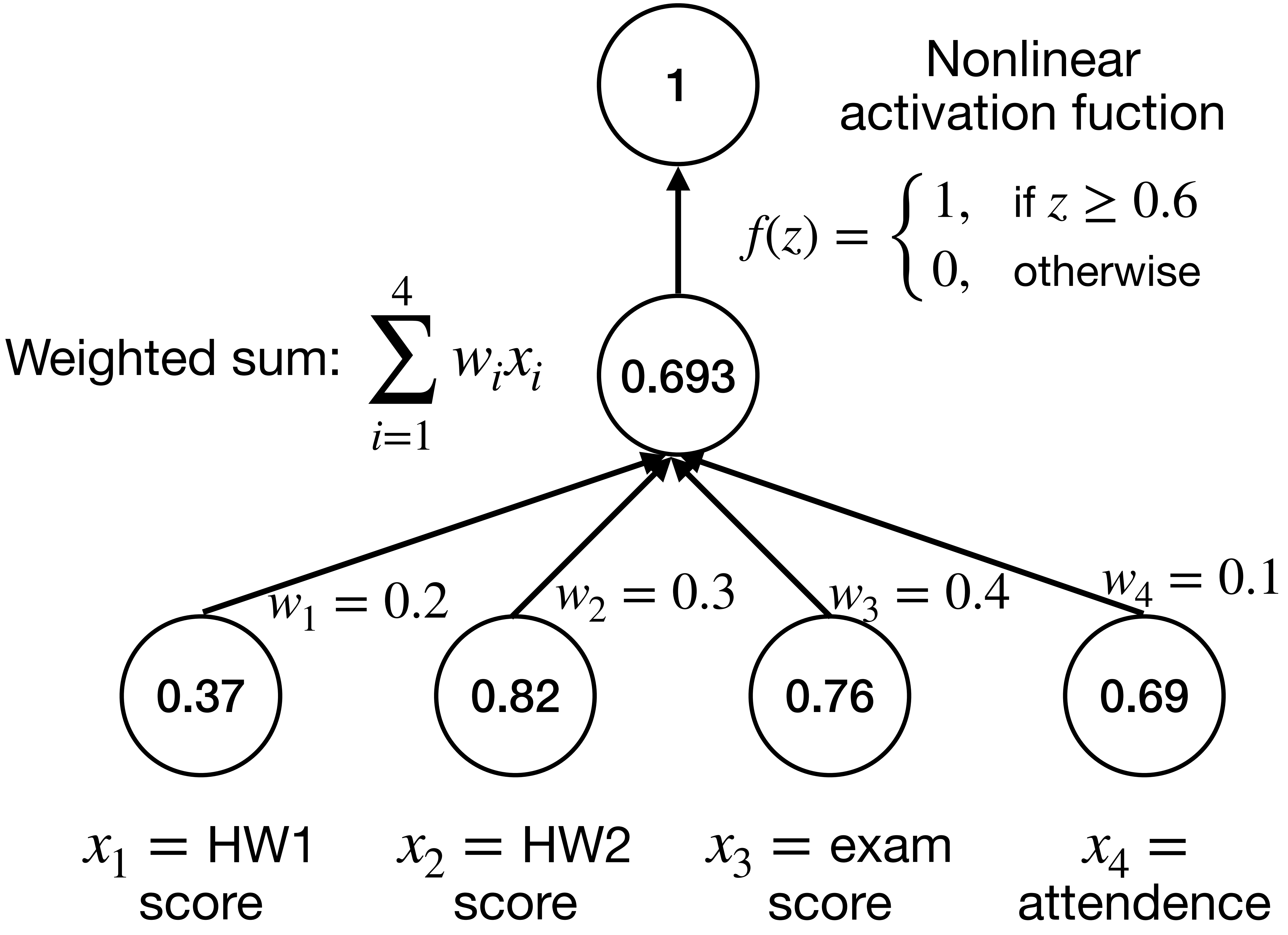
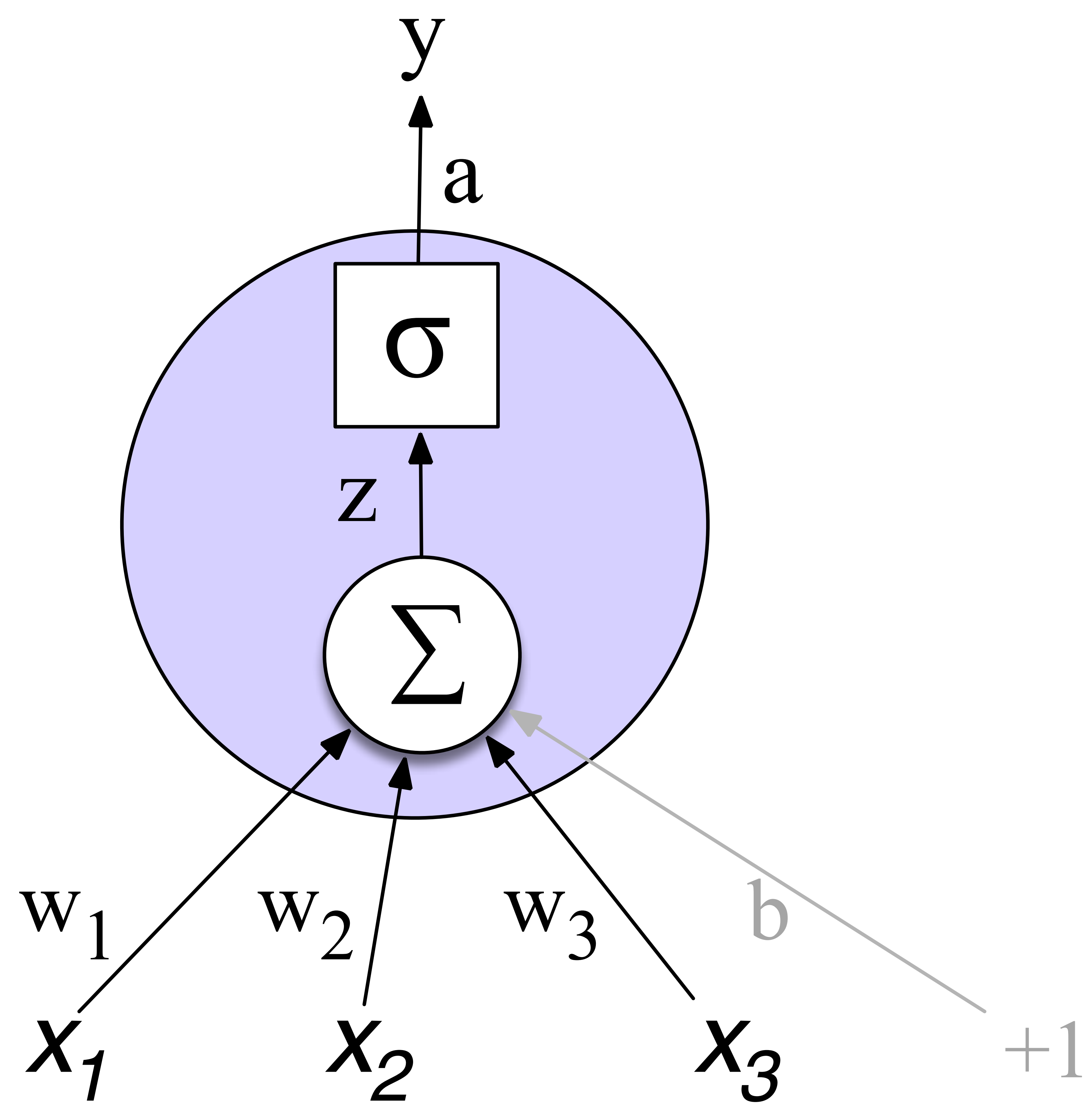


This becomes an Artificial Neuron!

a.k.a. Perceptron



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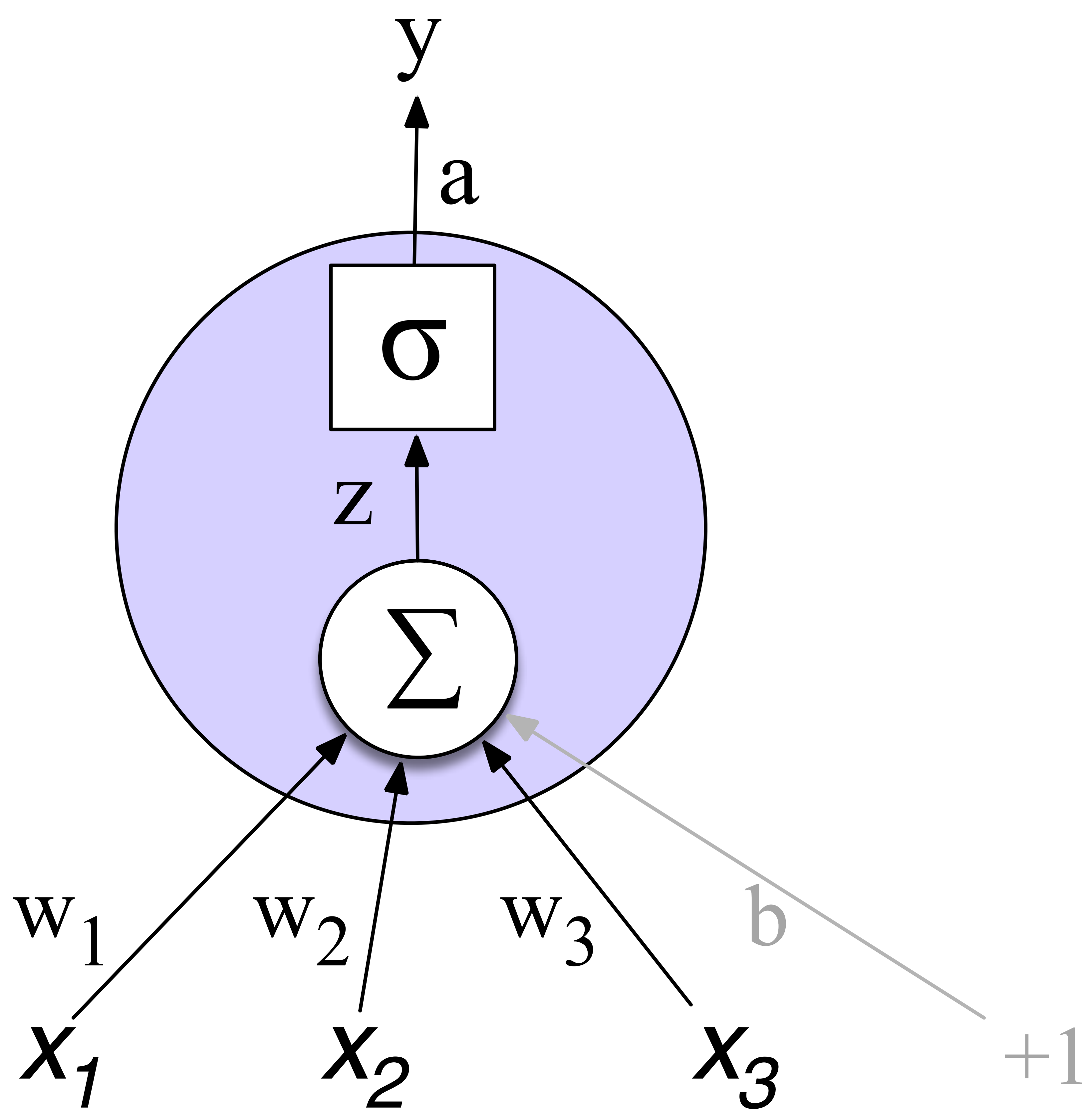

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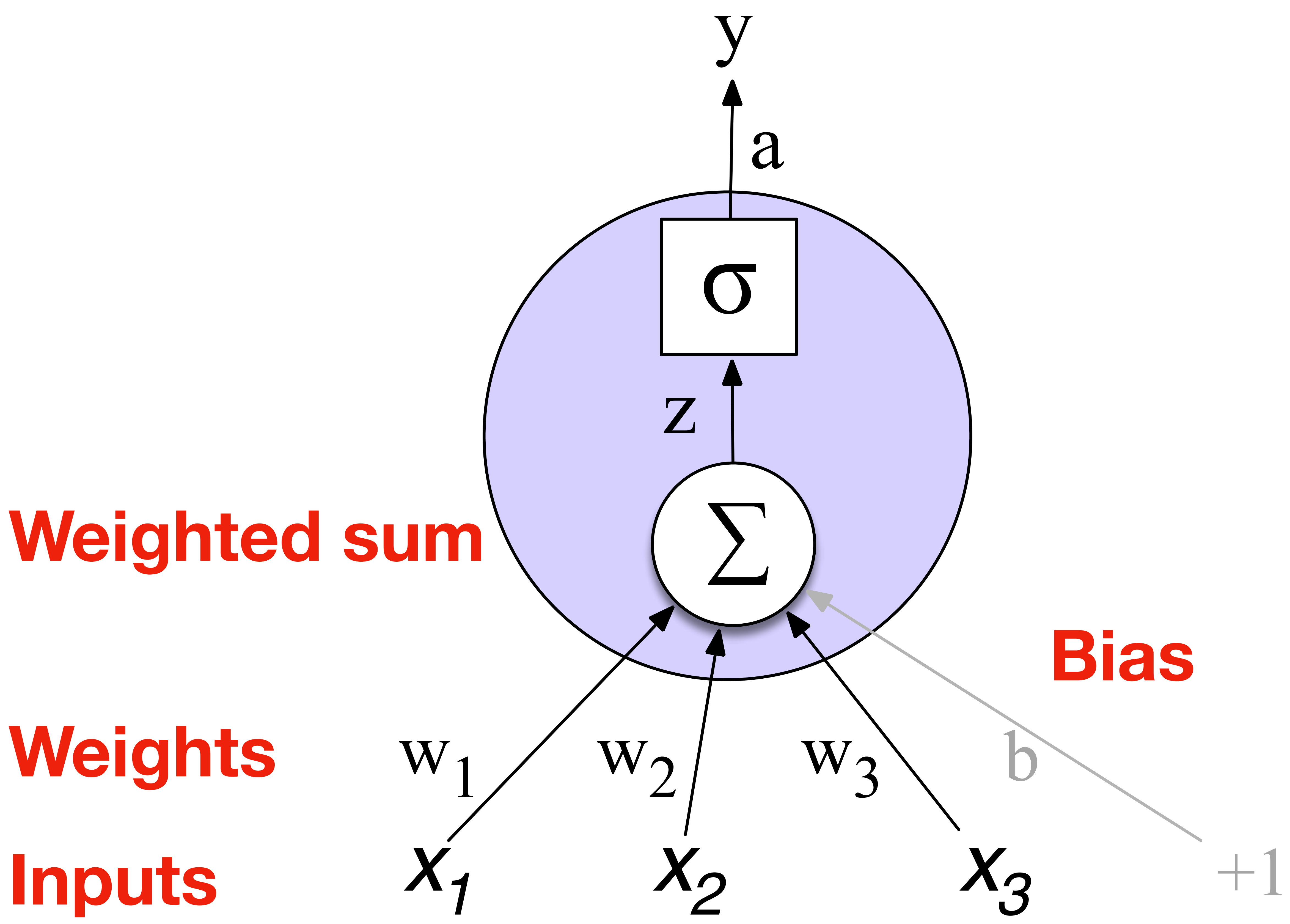

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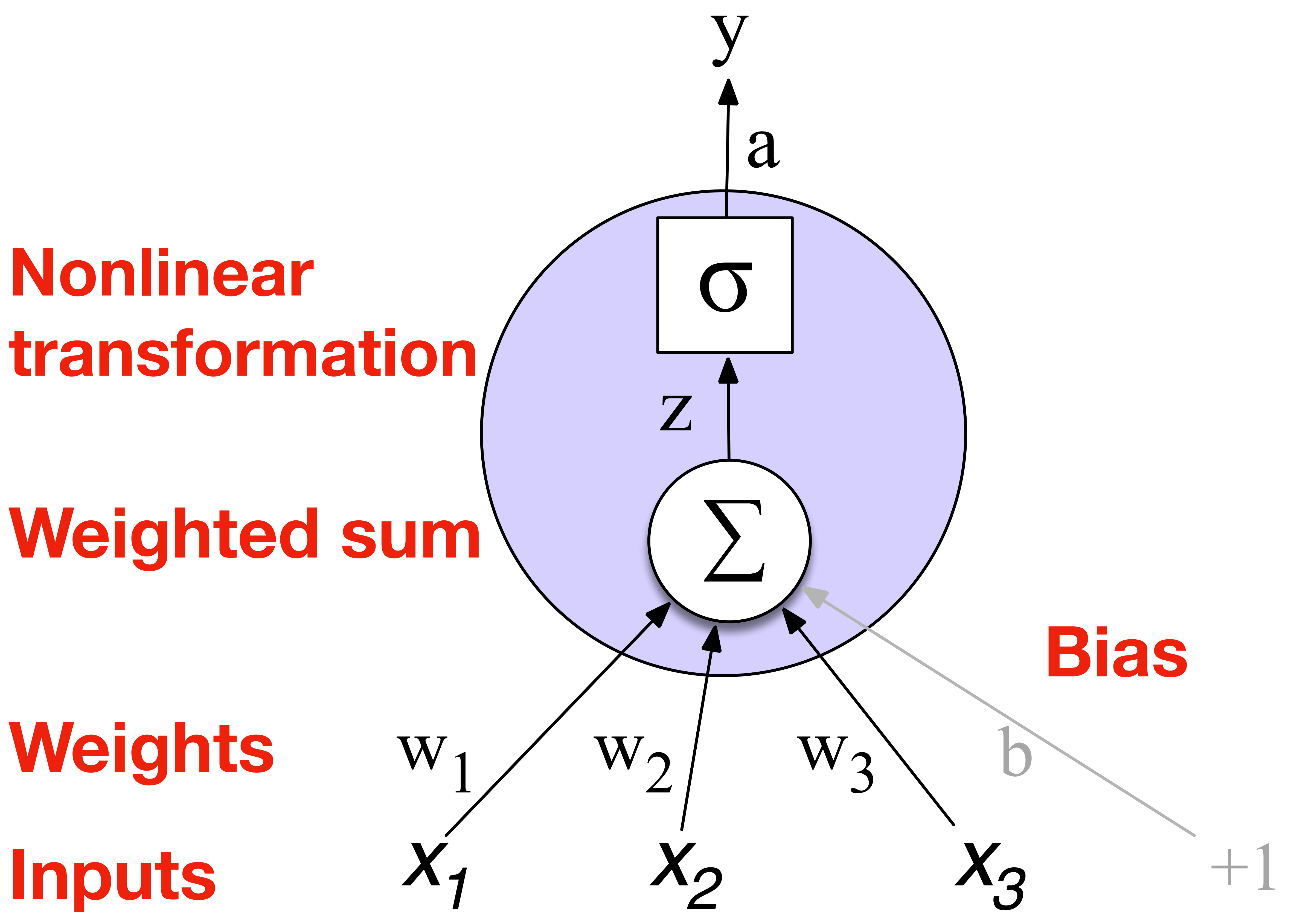

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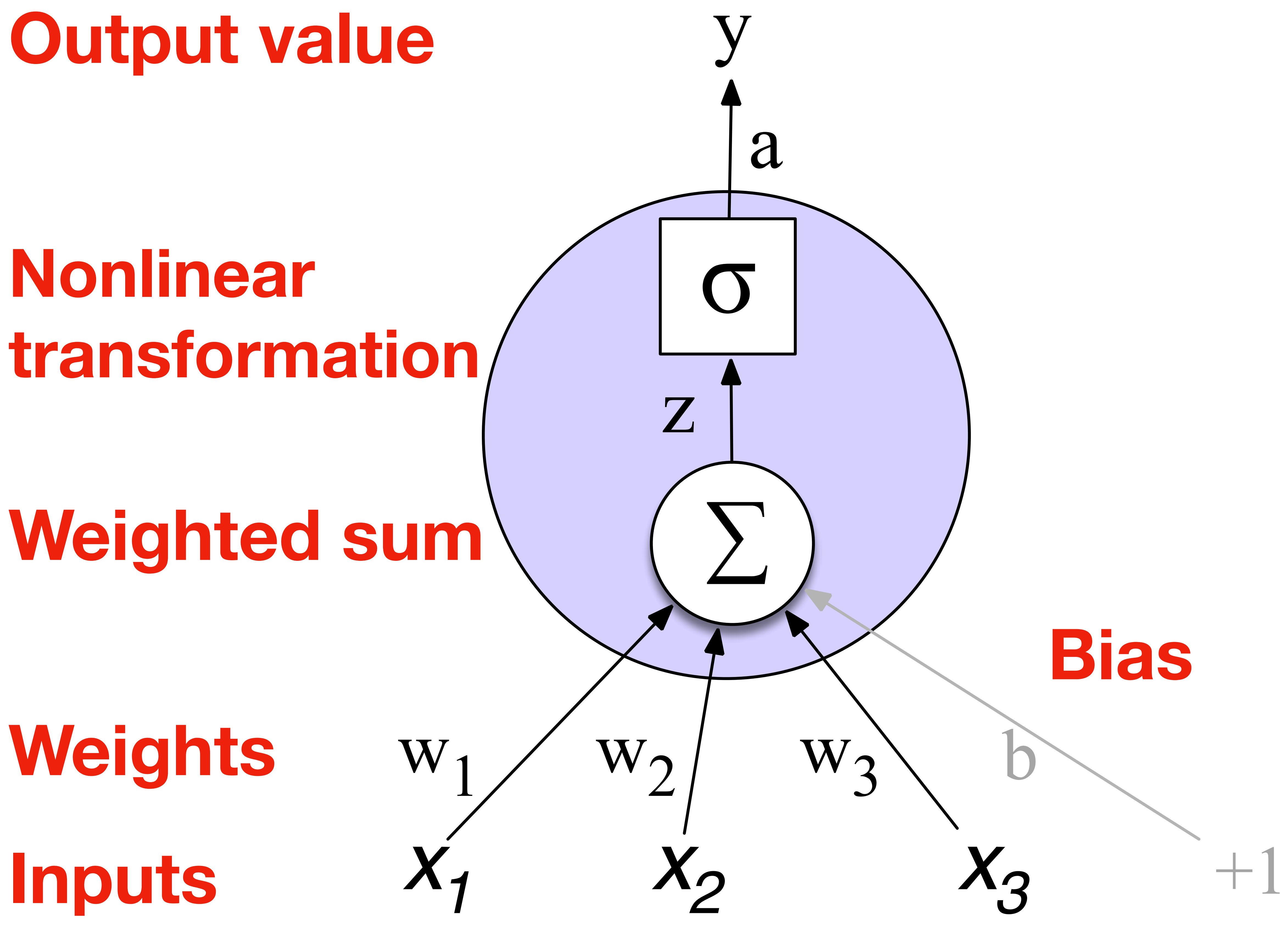

The Perceptron / Artificial Neuron

1958 - 1968

From Brain Neurons to the Perceptron

Inspiration: multiple inputs, weighted connections, one firing output.

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https://commons.wikimedia.org/w/index.php?curid=28761830;
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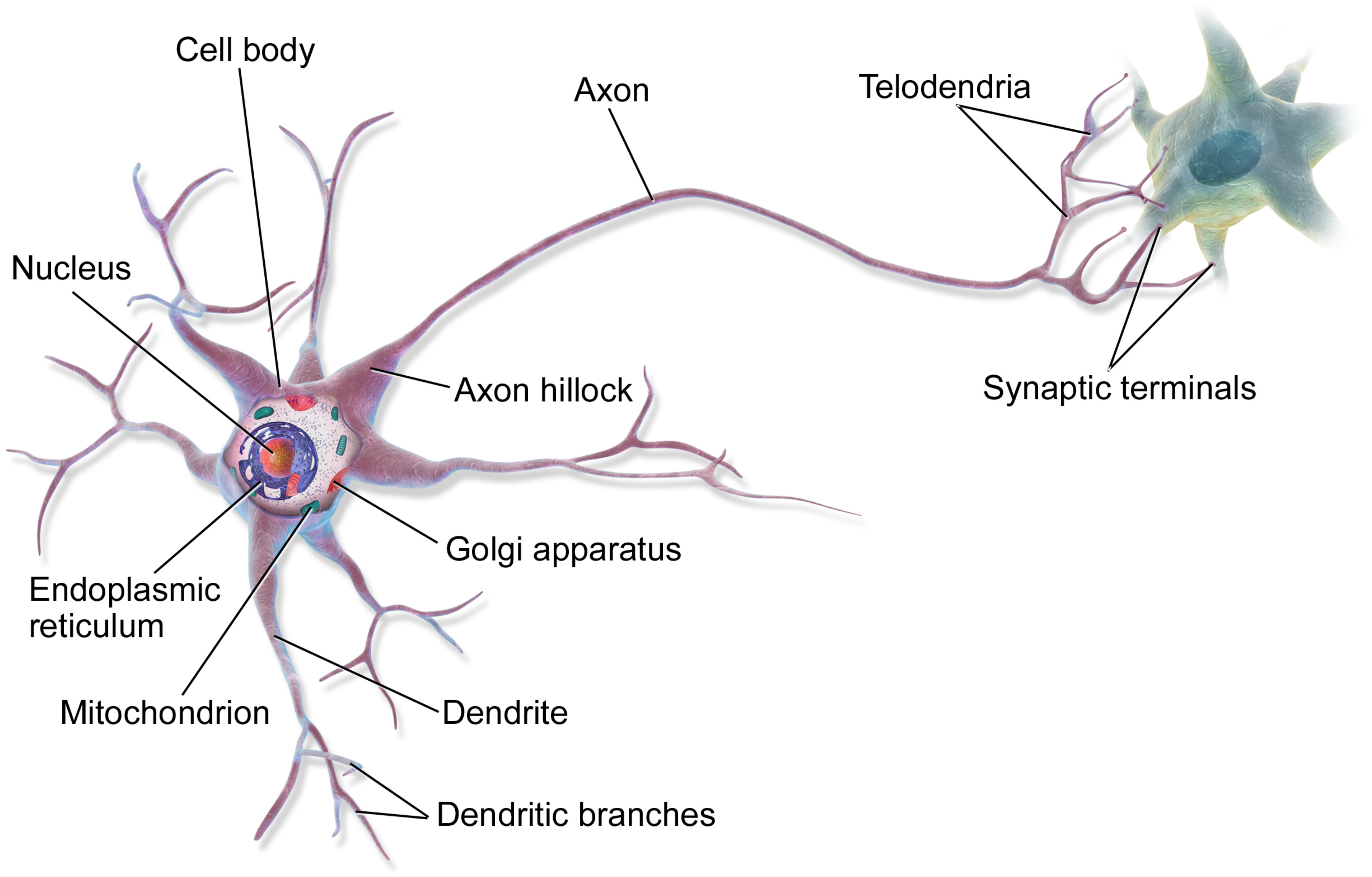

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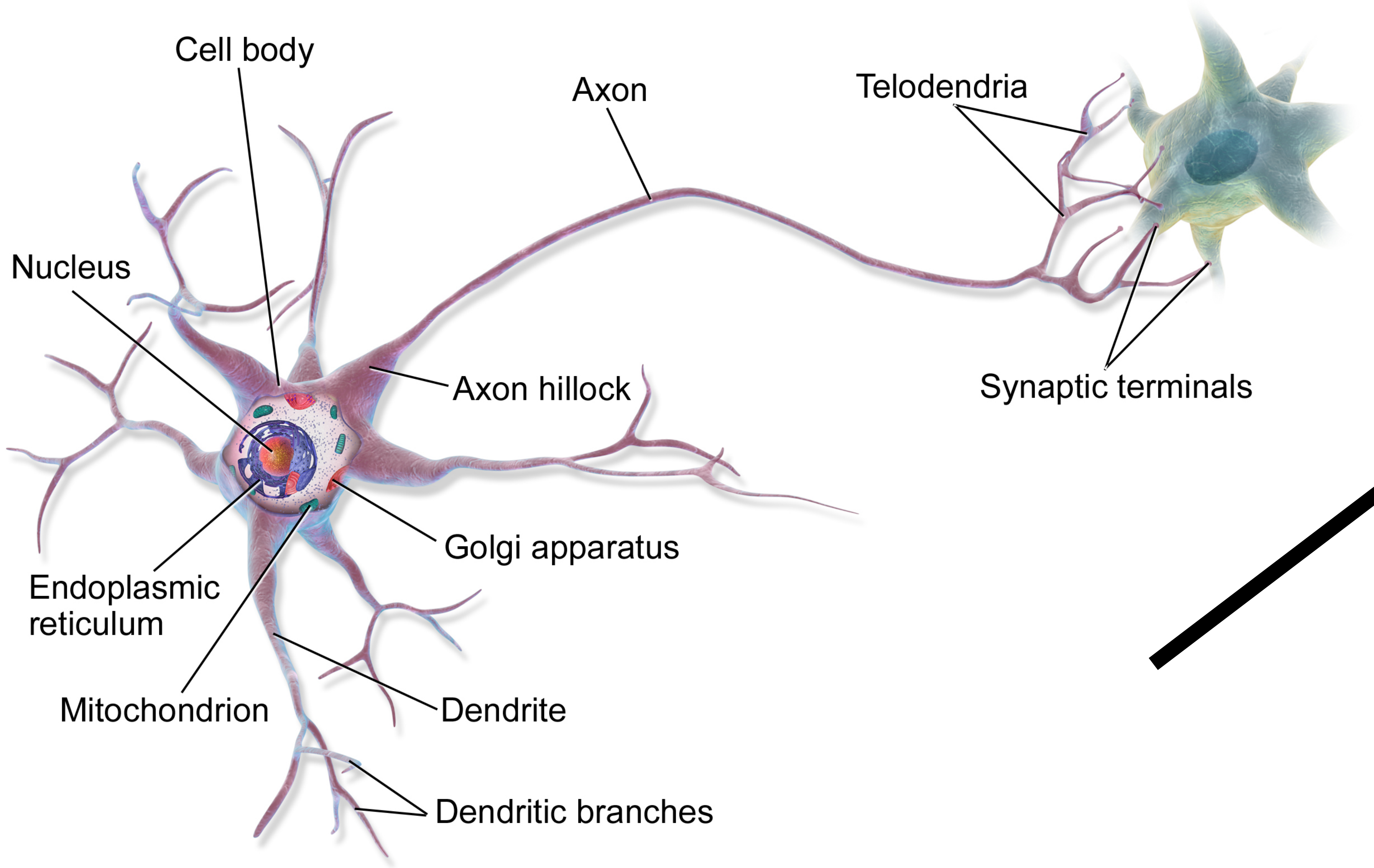

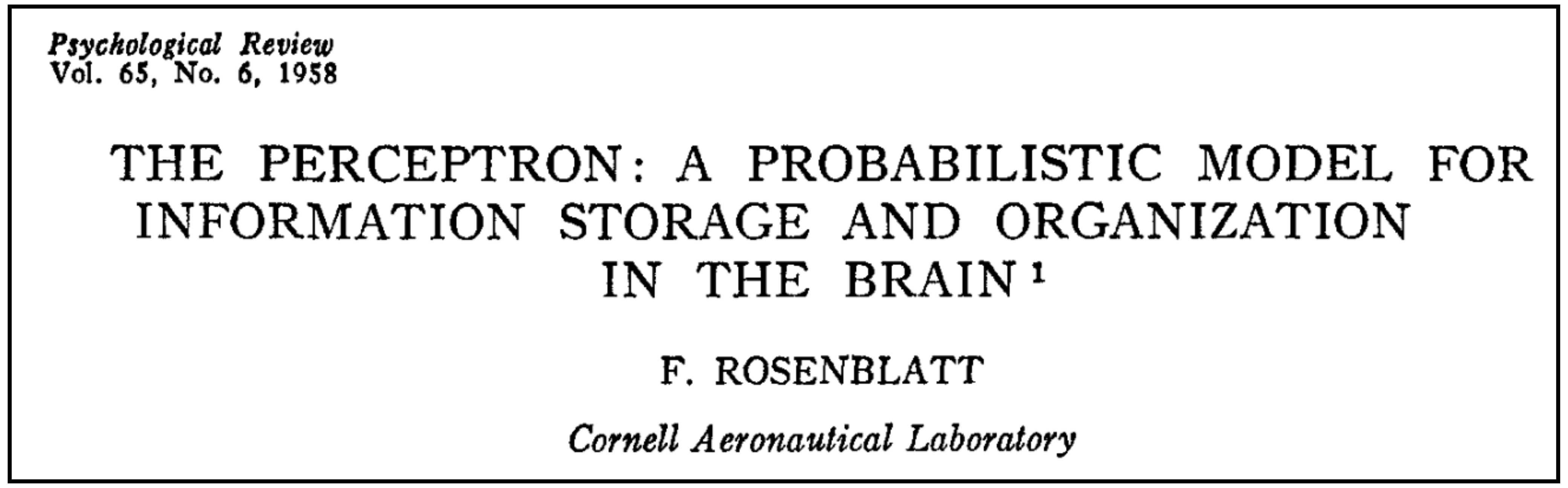
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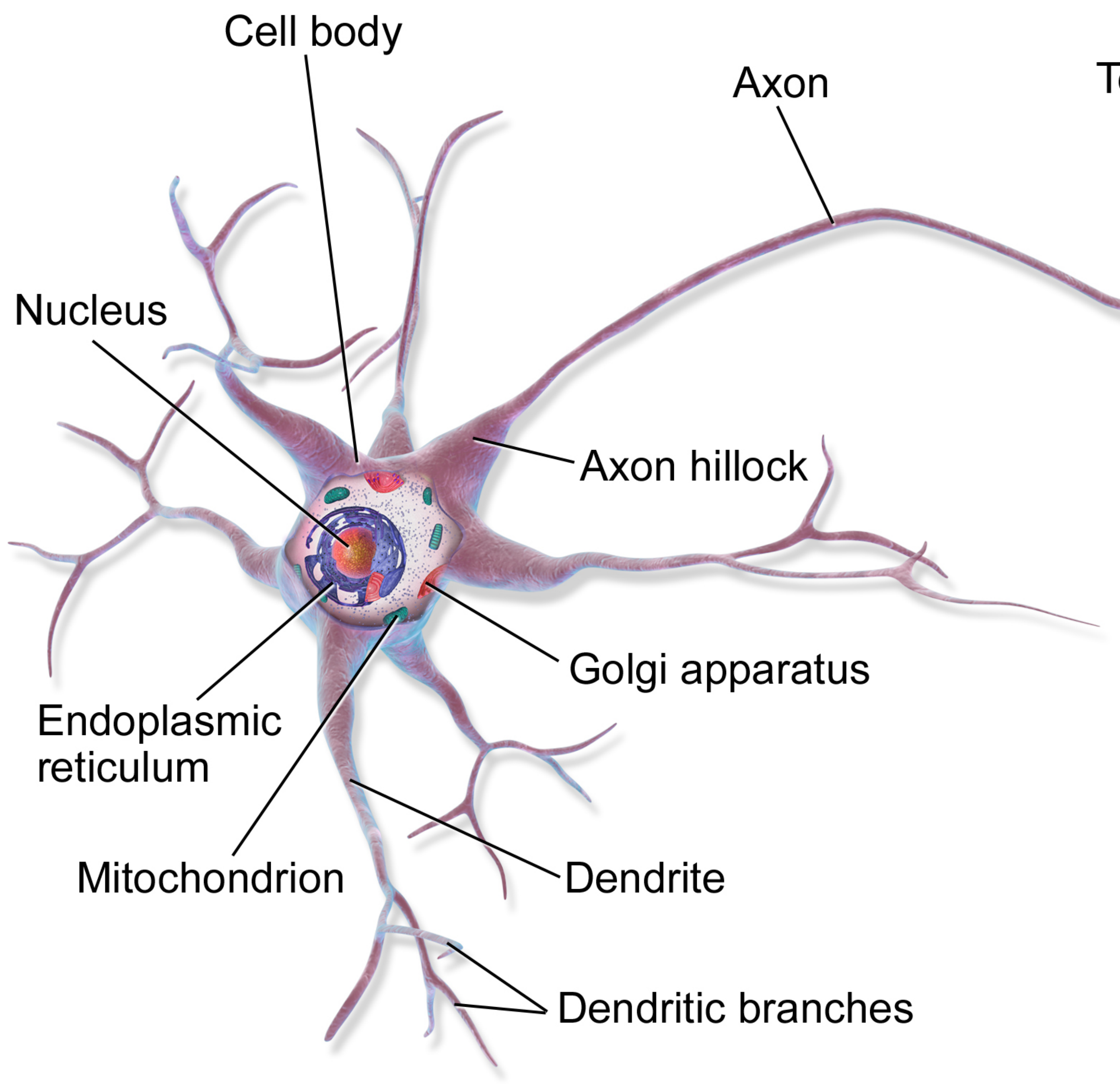

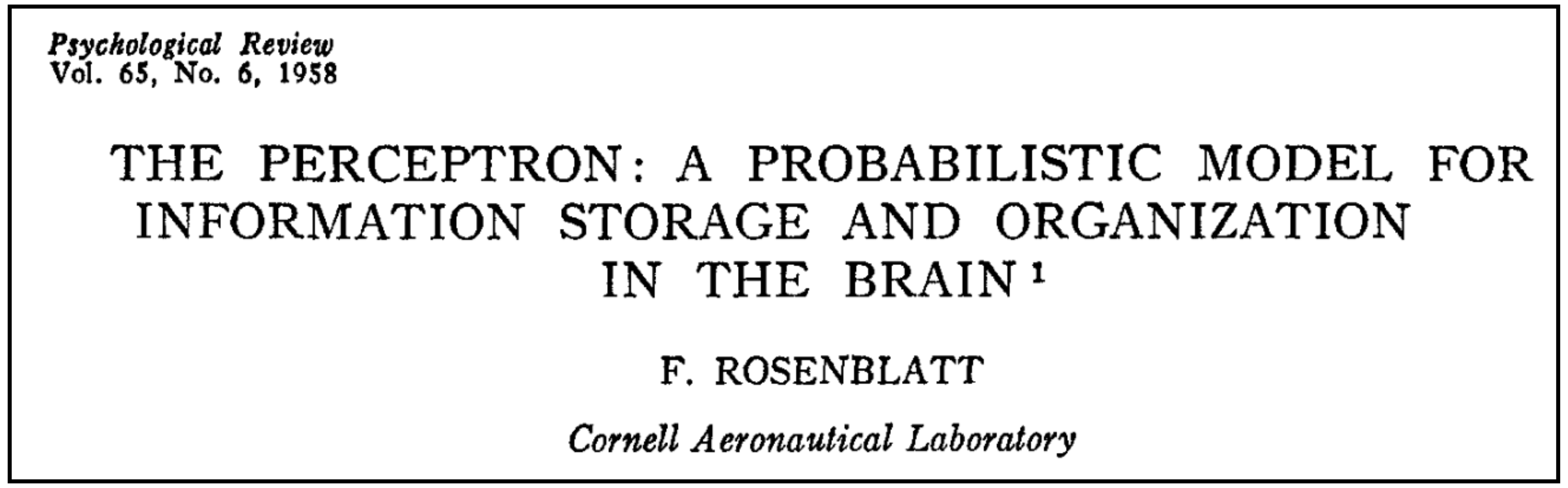
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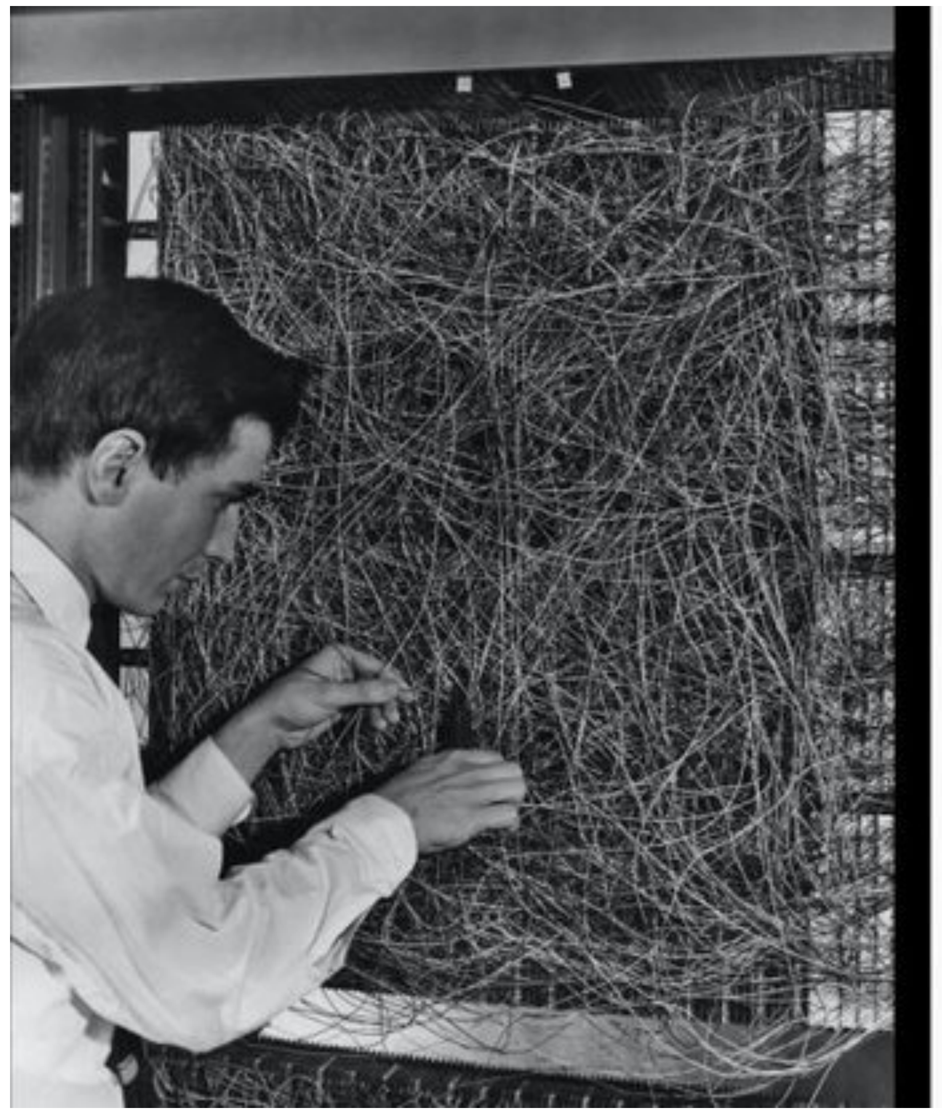

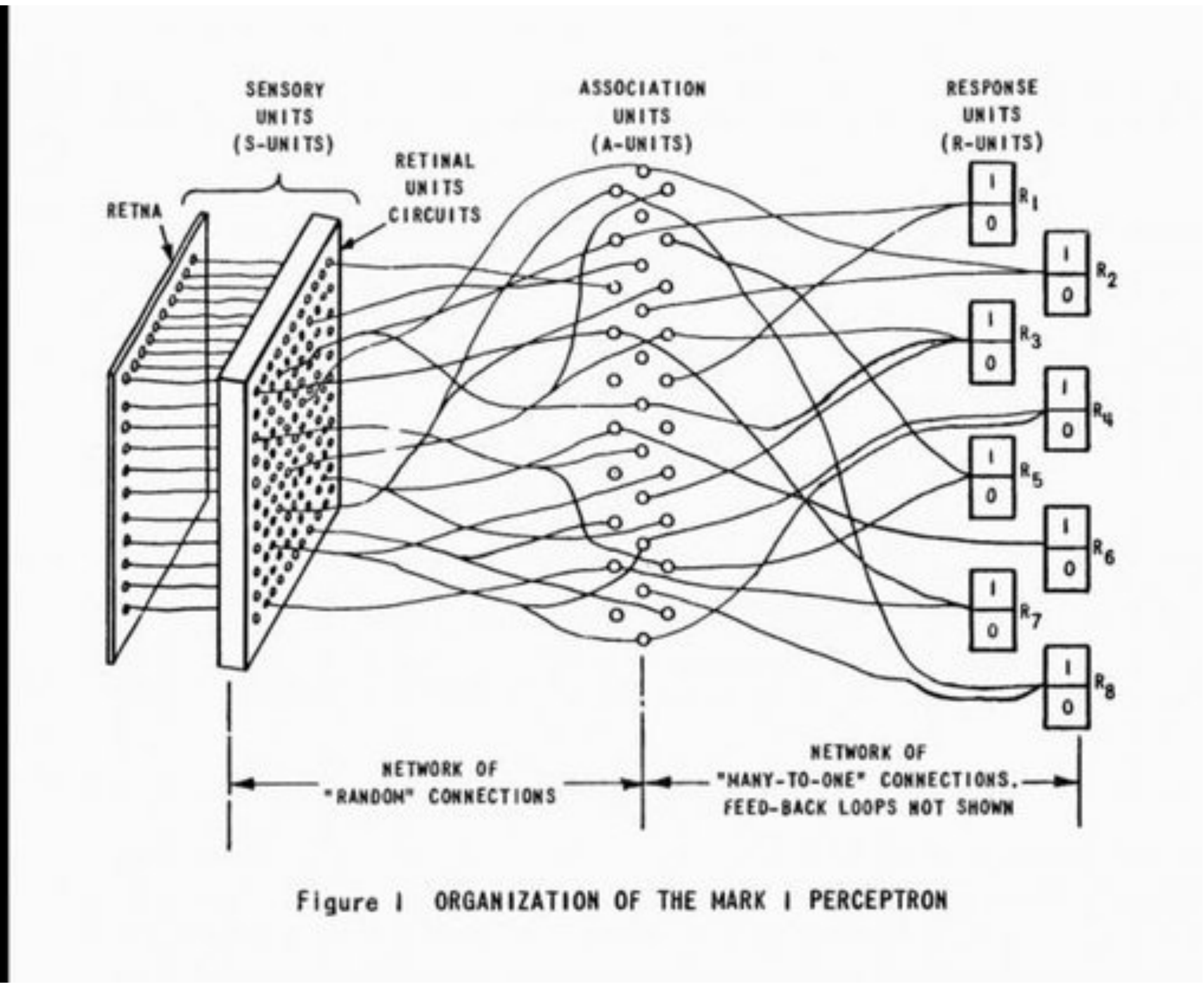

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- Therefore, $\sum_{i=1}^{n} w_i x_i = \mathbf{w} \cdot \mathbf{x}$, where $\cdot$ represents dot product (element-wise product between two vectors of the same size).
- In practice, we incorporate the bias term into the vector representation, where bias b always gets weight = 1.0;
    - This gives us $\mathbf{w} = [w_1, w_2, w_3, w_4, w_b] = [0.2, 0.3, 0.5, 0.1, 1.0]$

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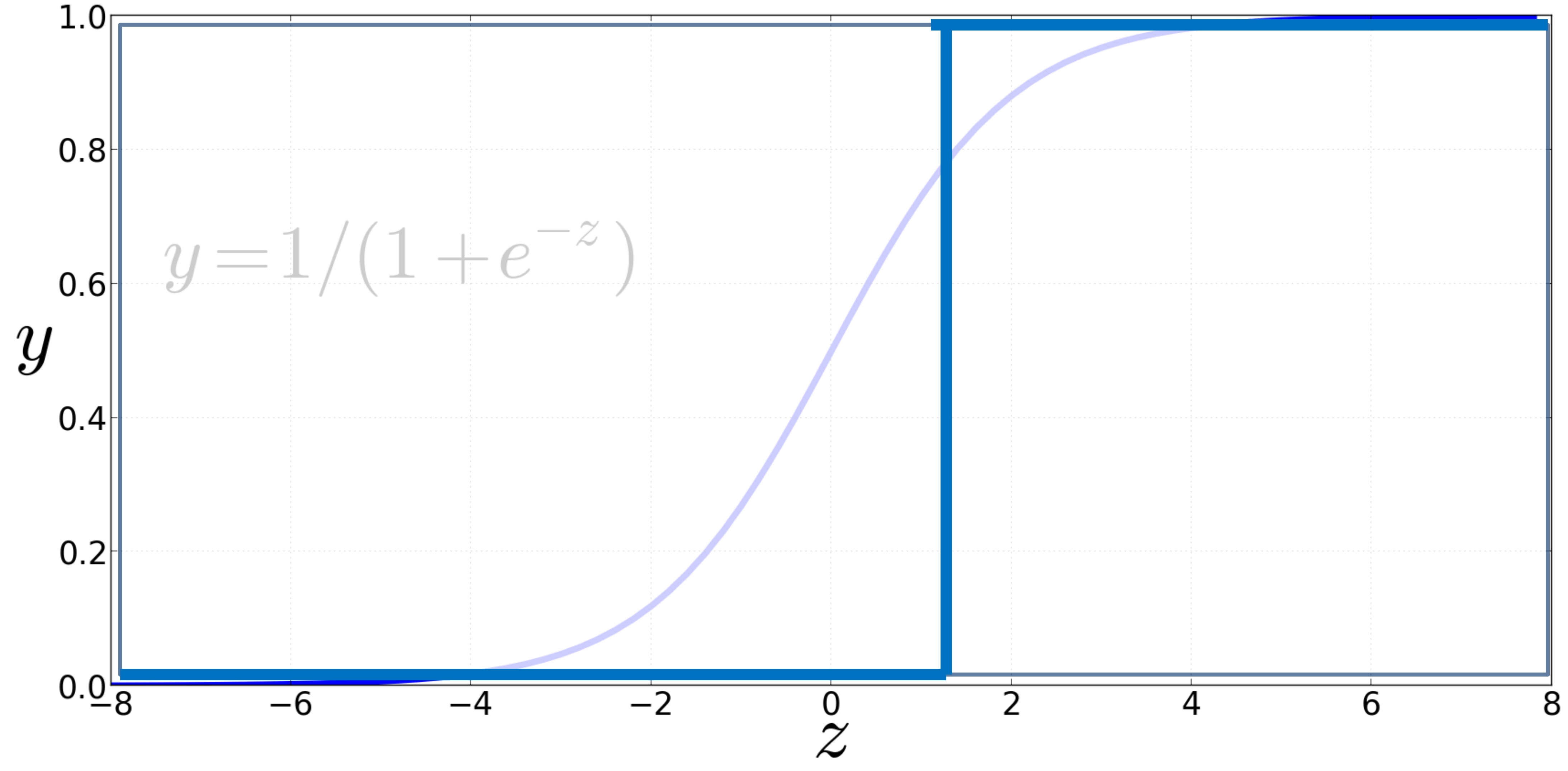

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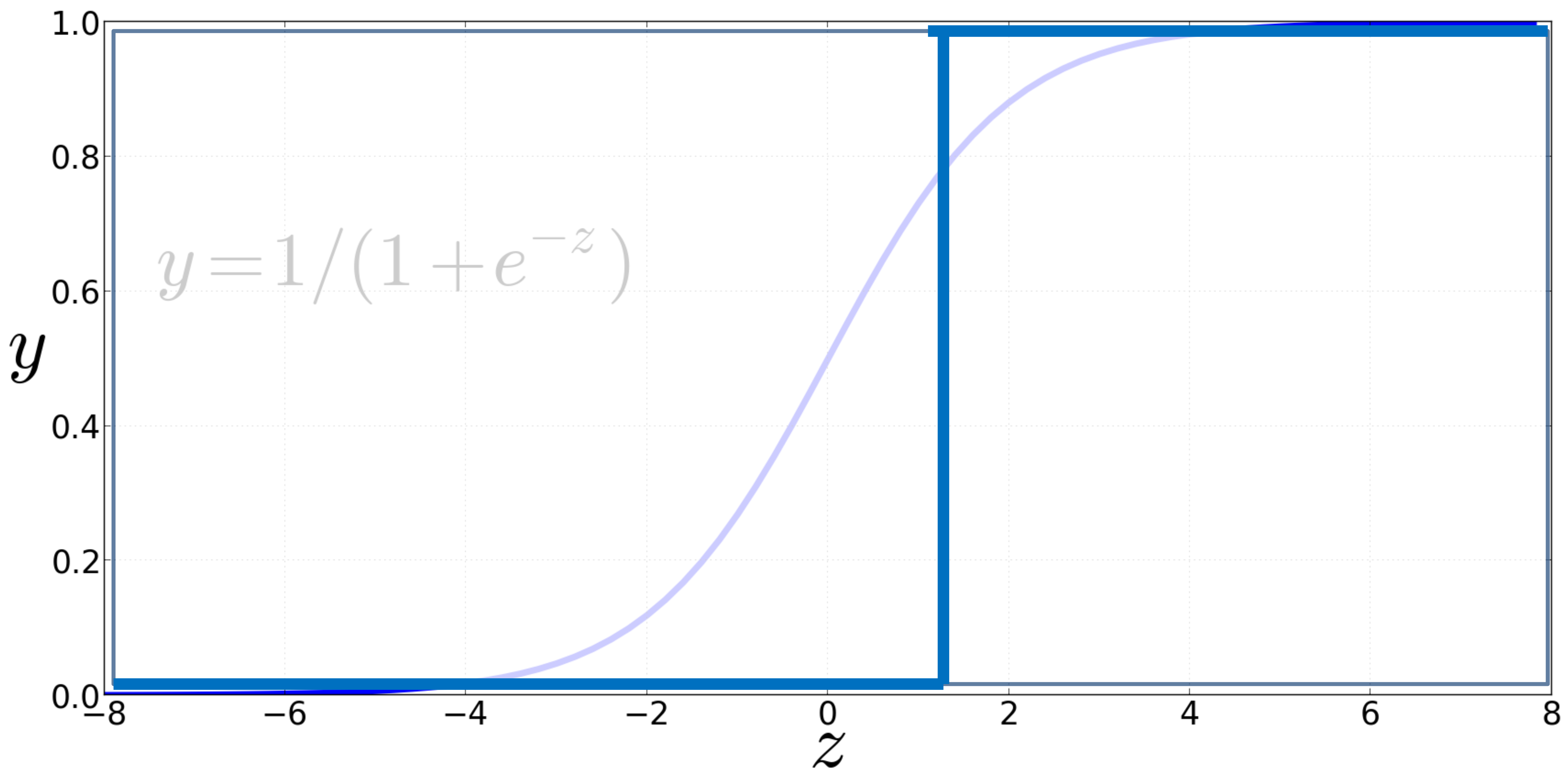
Intuition: nonlinearity enables the model to represent relations beyond linear functions (i.e., cannot be represented by matrix operations)!

* *will come back to it later*



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Choose activation functions wisely given your problem setting

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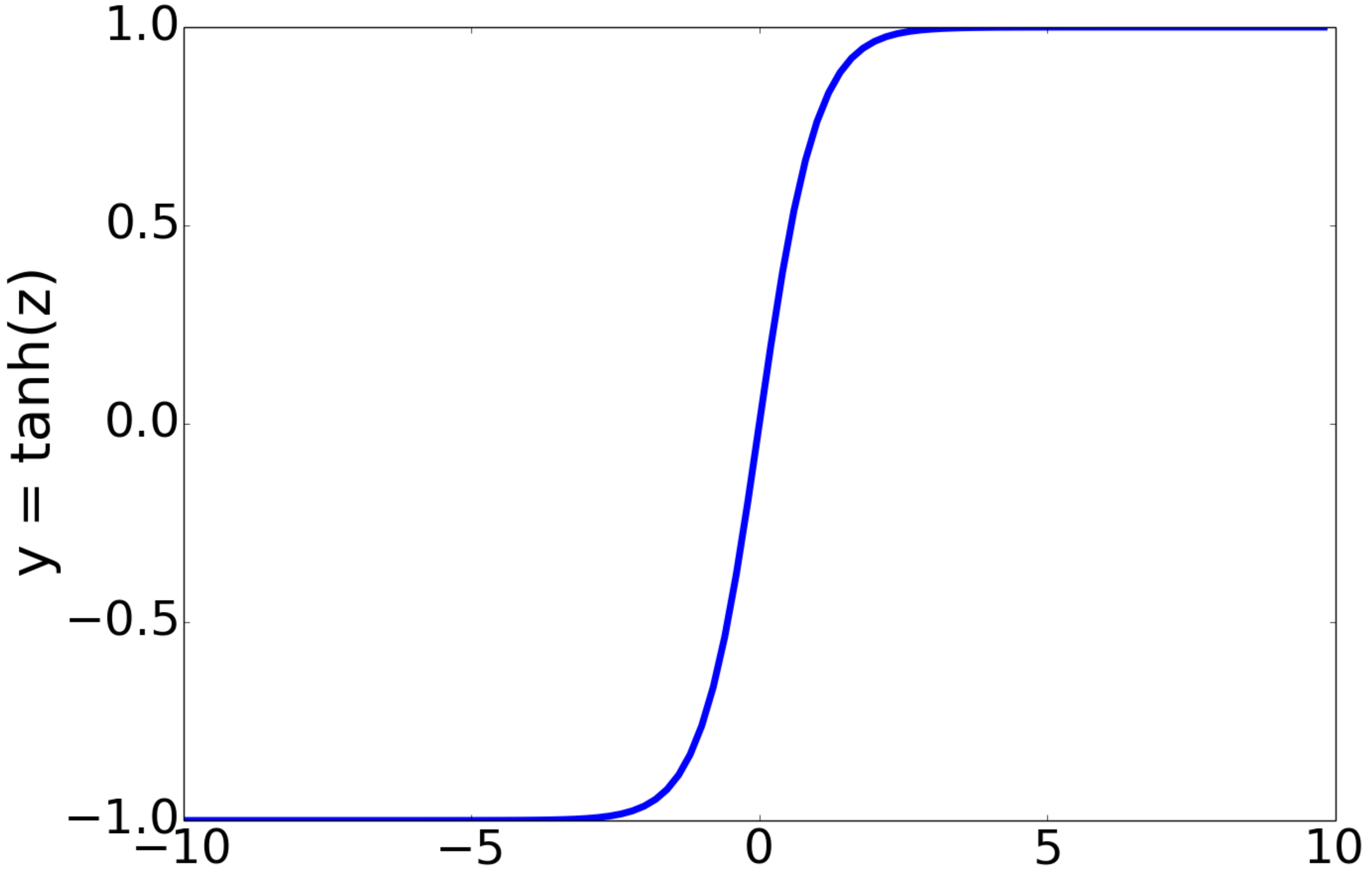

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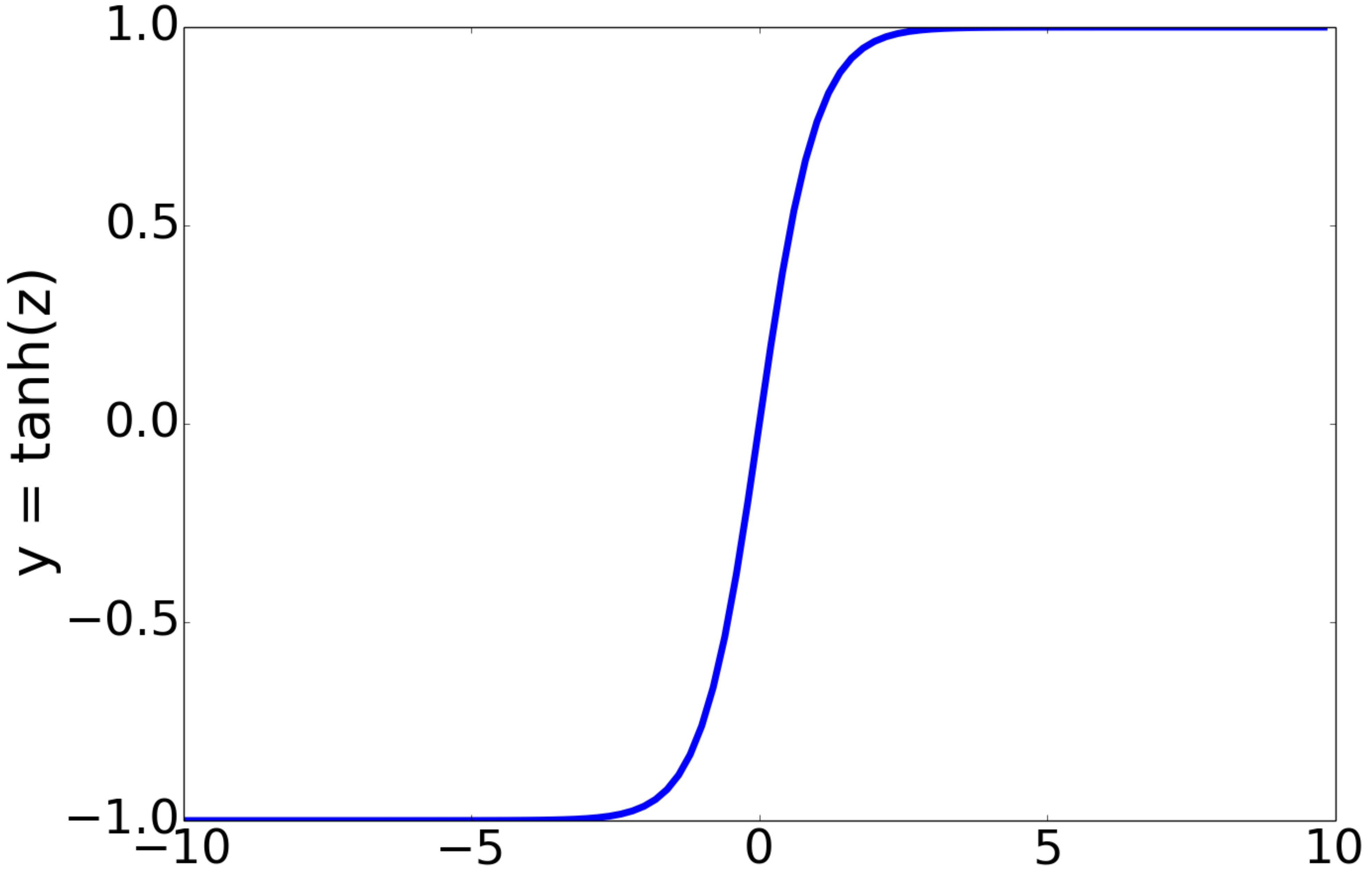
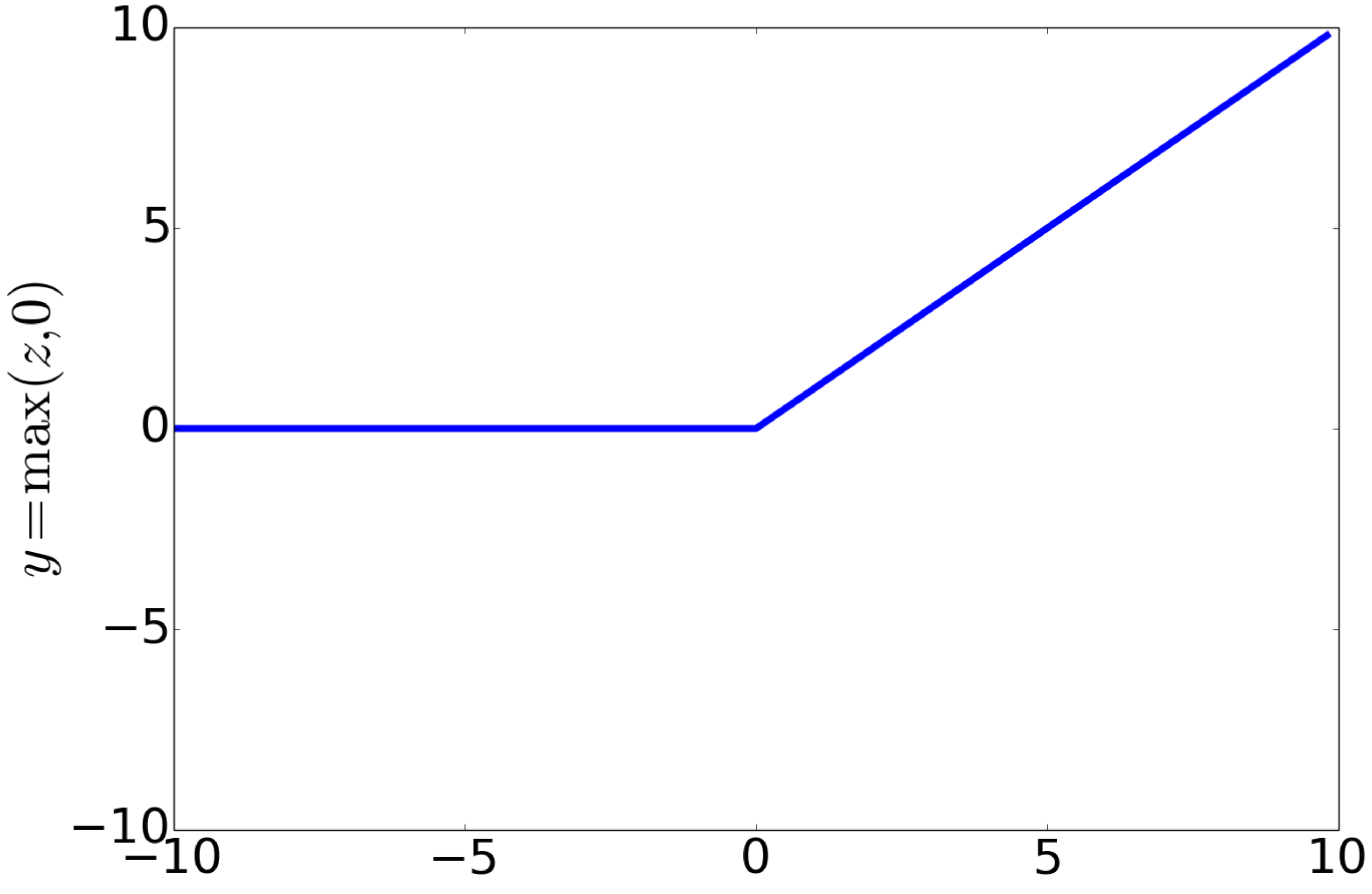


tanh

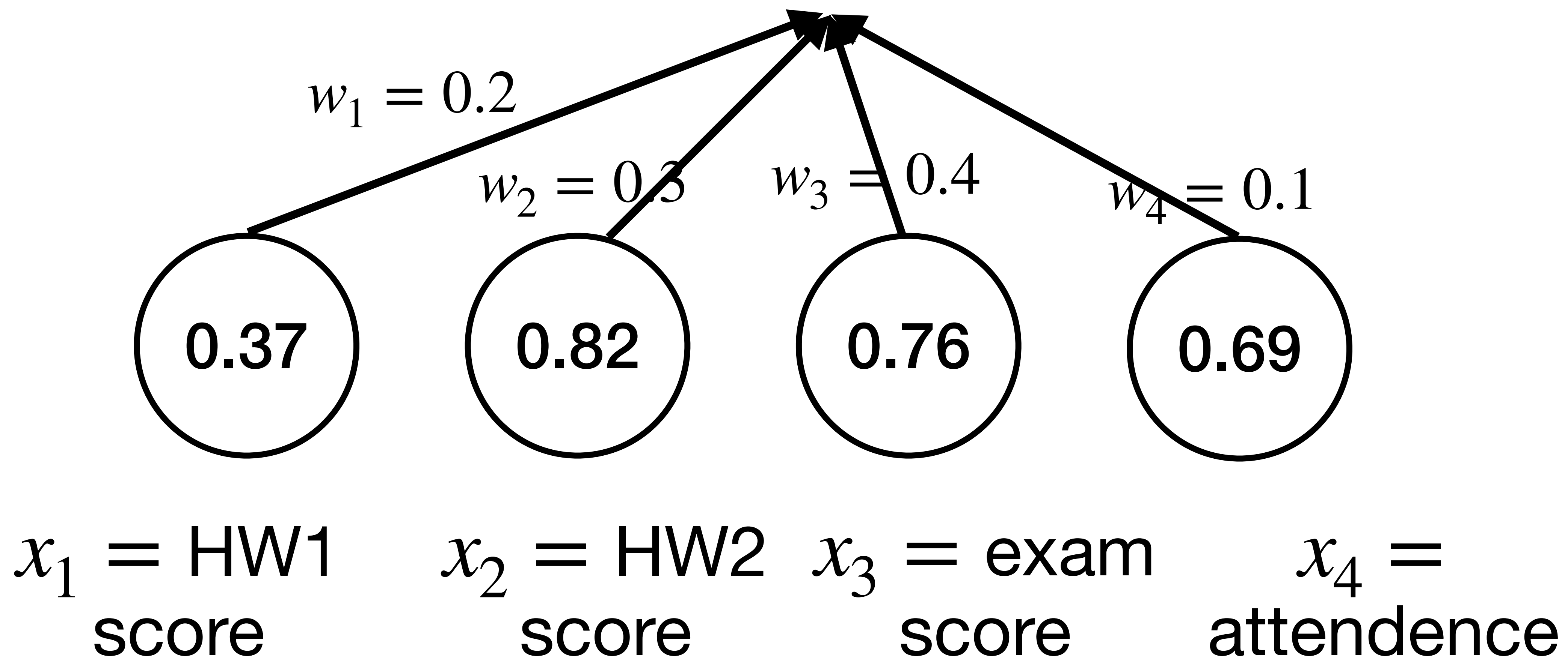
$$\text{ReLu}(z) = max(z,\ 0)$$



ReLU
(Rectified Linear Unit)

An Example Computation

Back to the grading example



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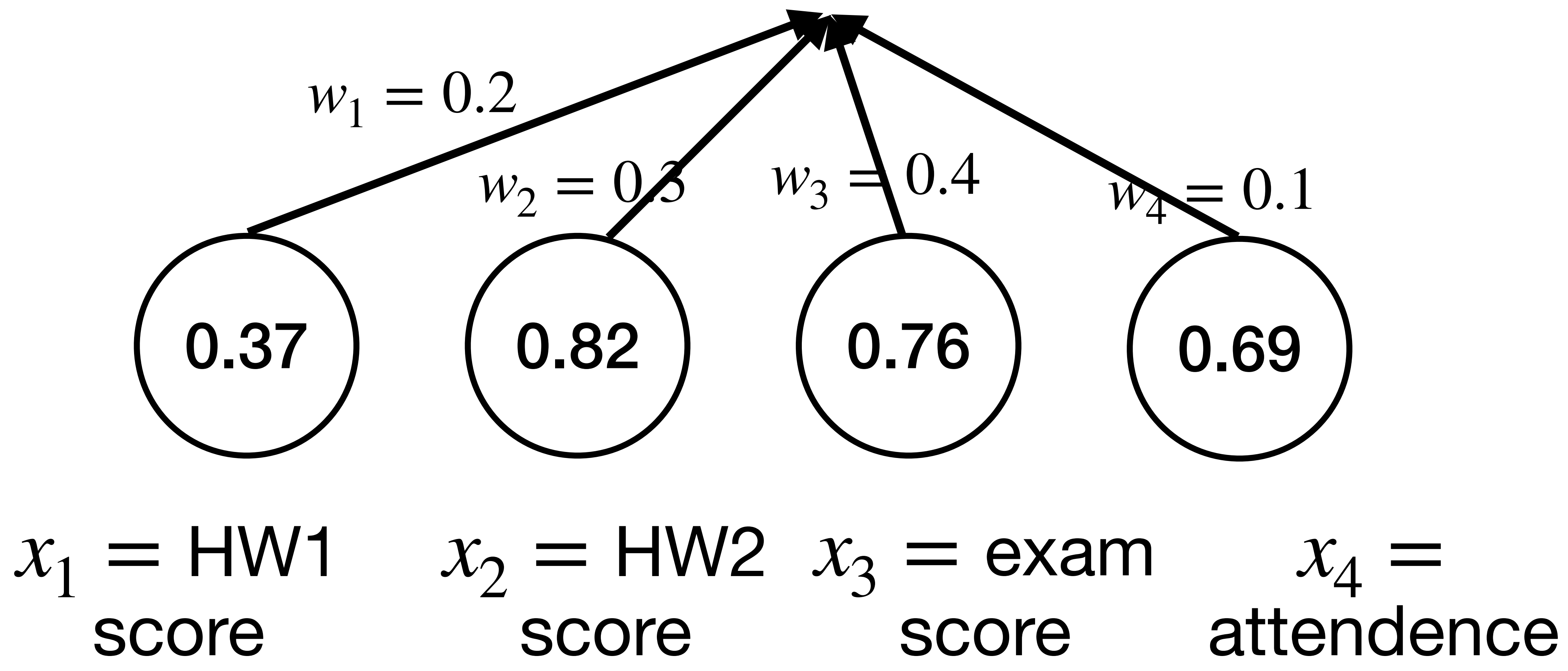

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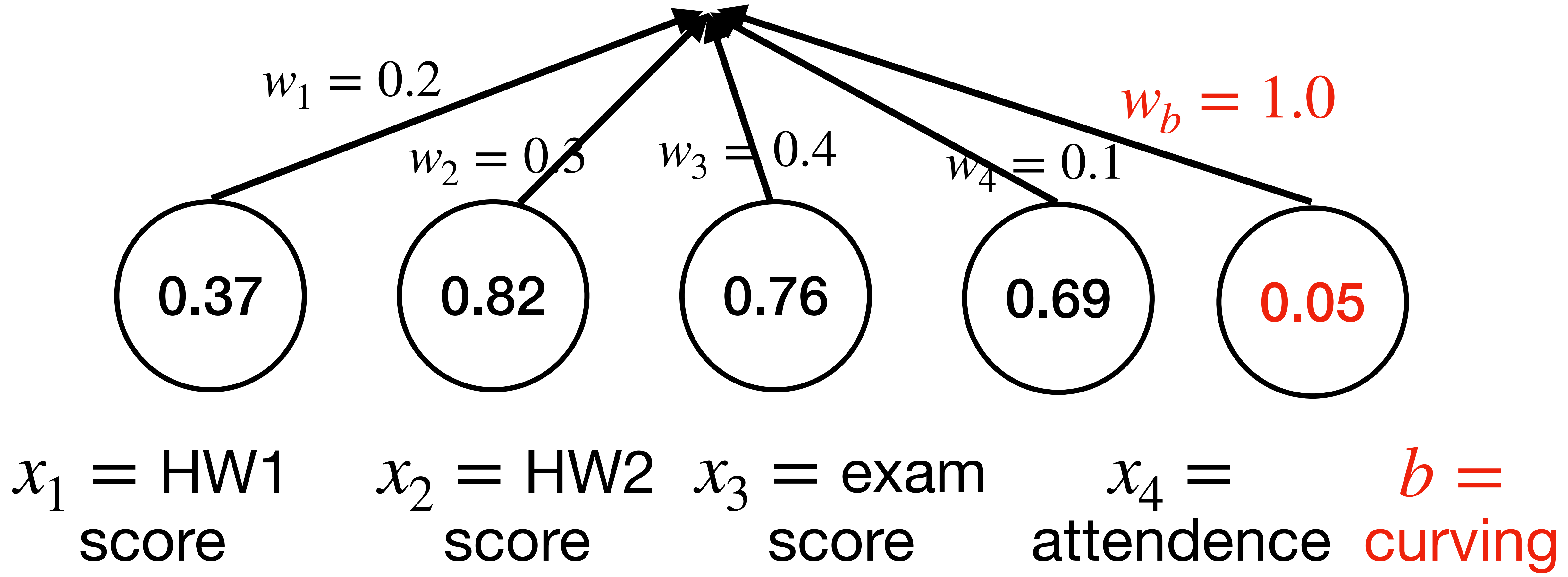

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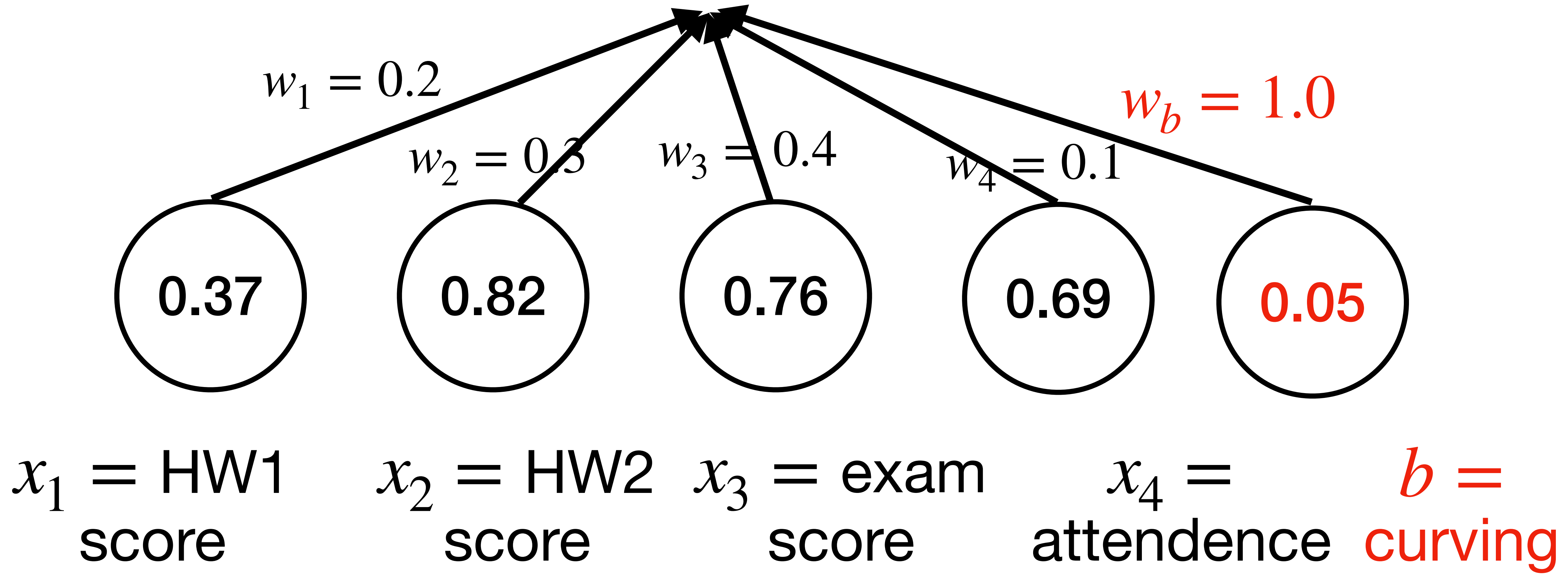

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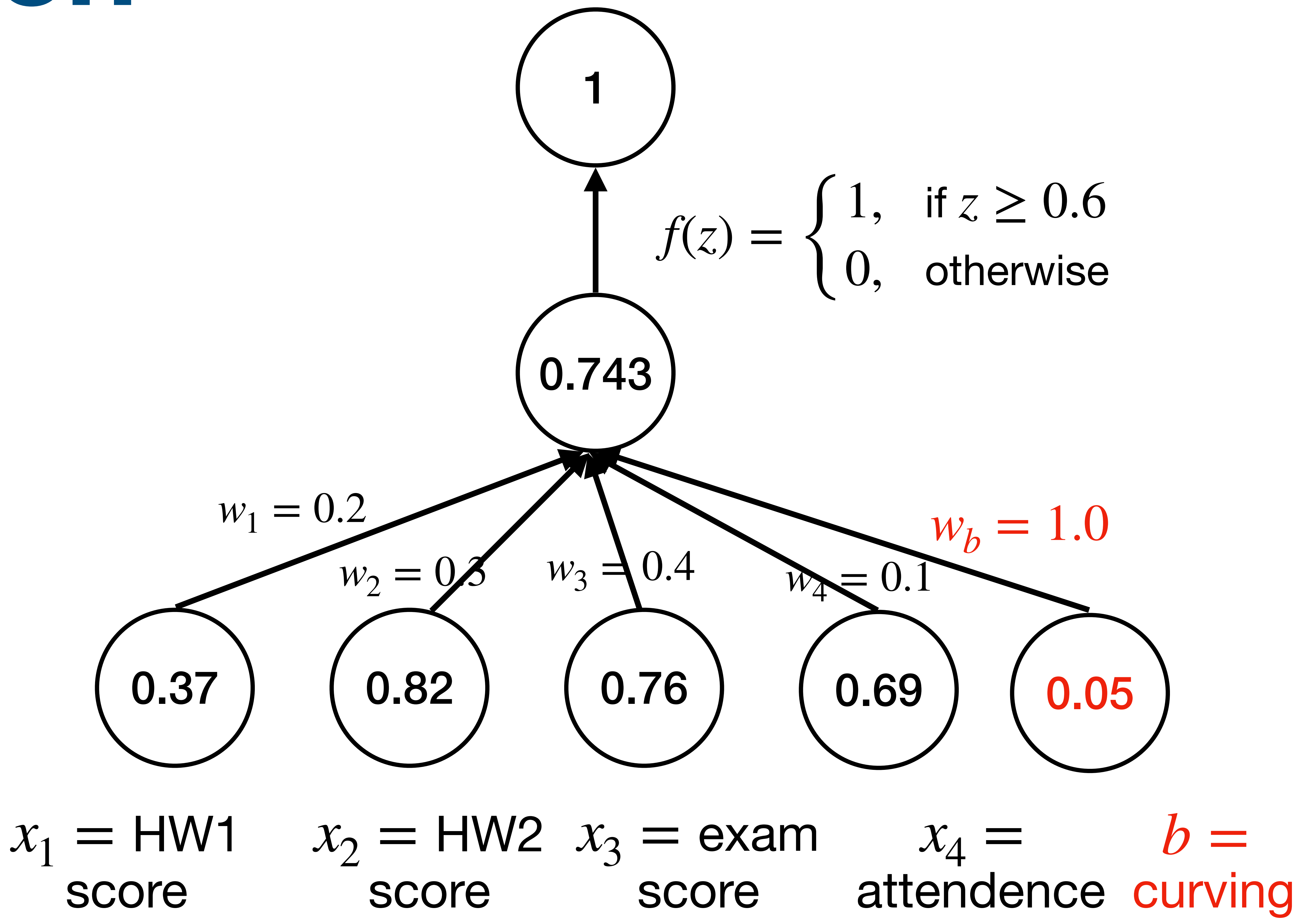
$$= f(0.2 \times 0.37 + 0.3 \times 0.82 + 0.4 \times 0.76 + 0.1 \times 0.69 + 1.0 \times 0.05)$$
$$= f(0.743) = Boolean(0.743 \geq 0.6) = 1$$



Wait…. Where do weights come from?

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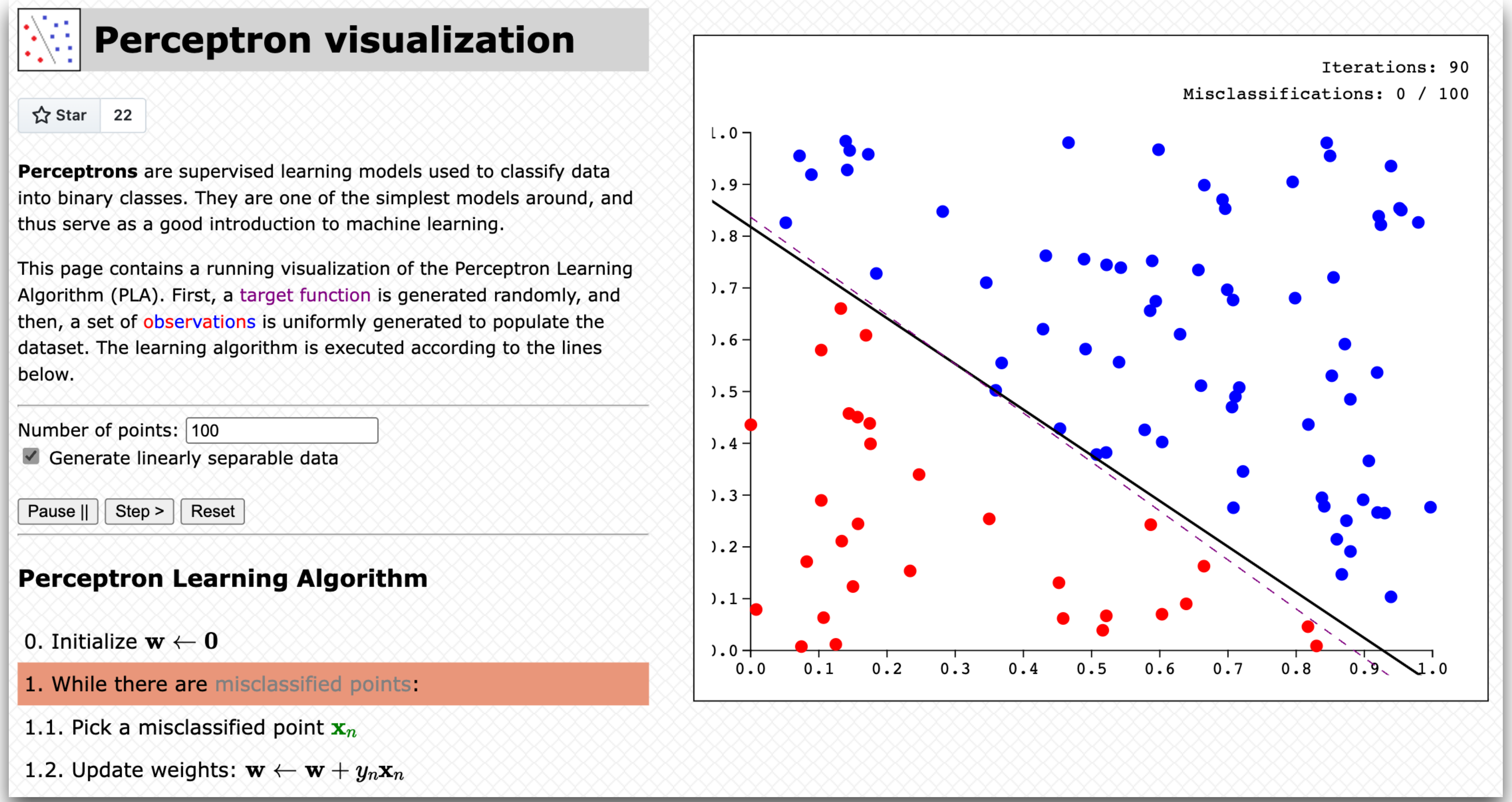
- **Intuition (very important!)**:
  - If I made an error ($\hat{y}$ differs from y, so that $(y - \hat{y}) = \pm 1$), then update each weight w_i with the amount ηx_i towards the correct direction (sign of $y - \hat{y}$).
  - Each value of x_i determines, for this example, how much is w_i *faulty / responsible* for this mis-classification!

A Real-Time Illustration

Perceptron is really “perceiving and adapting” — aka learning!



Interium Summary 1

Logistic Regression → Artifical Neuron / Perceptron → Learning

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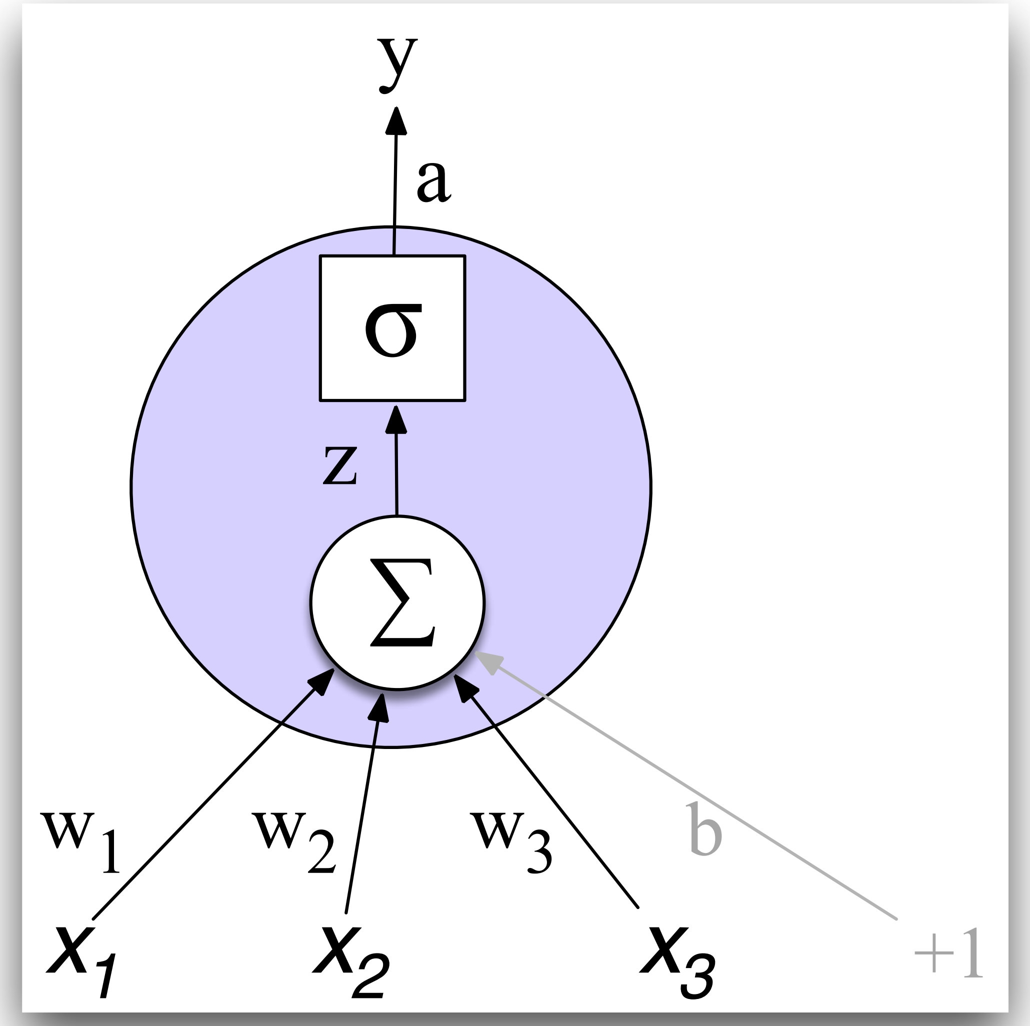

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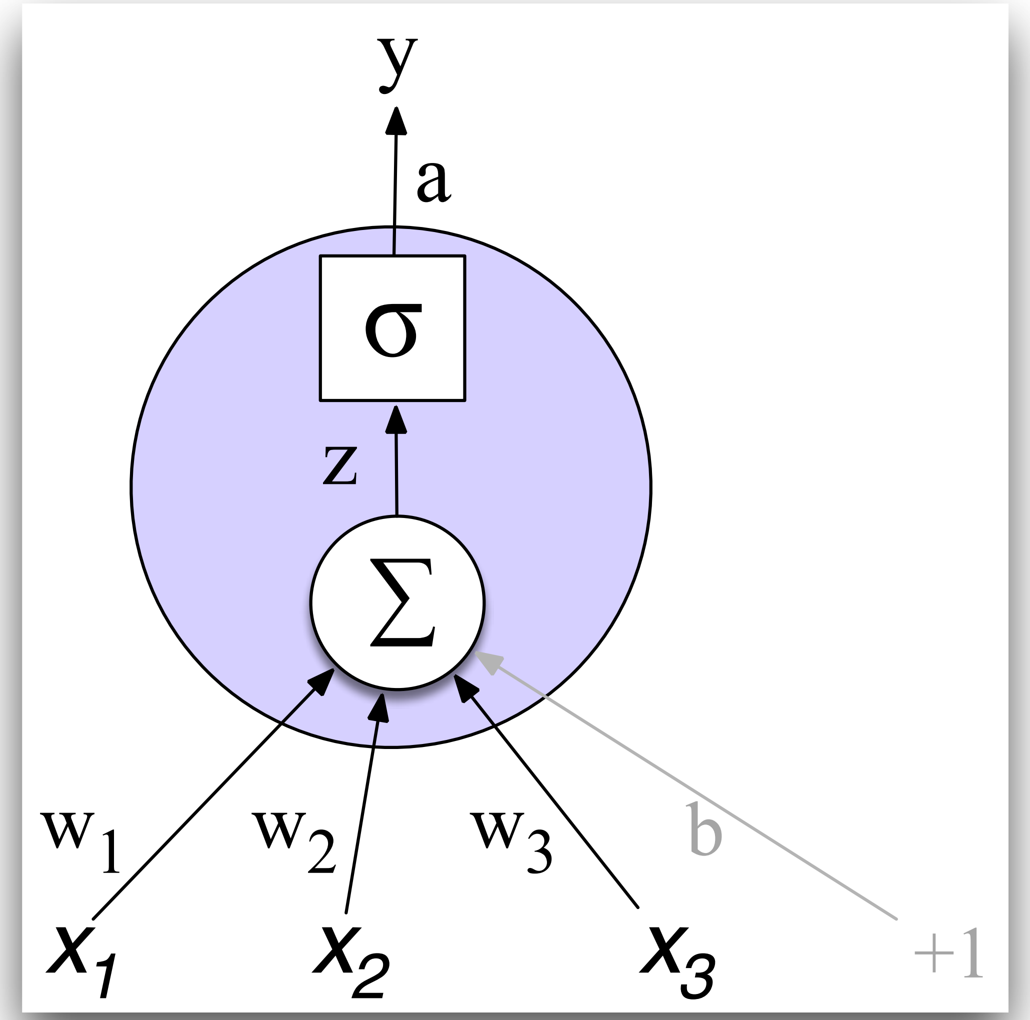

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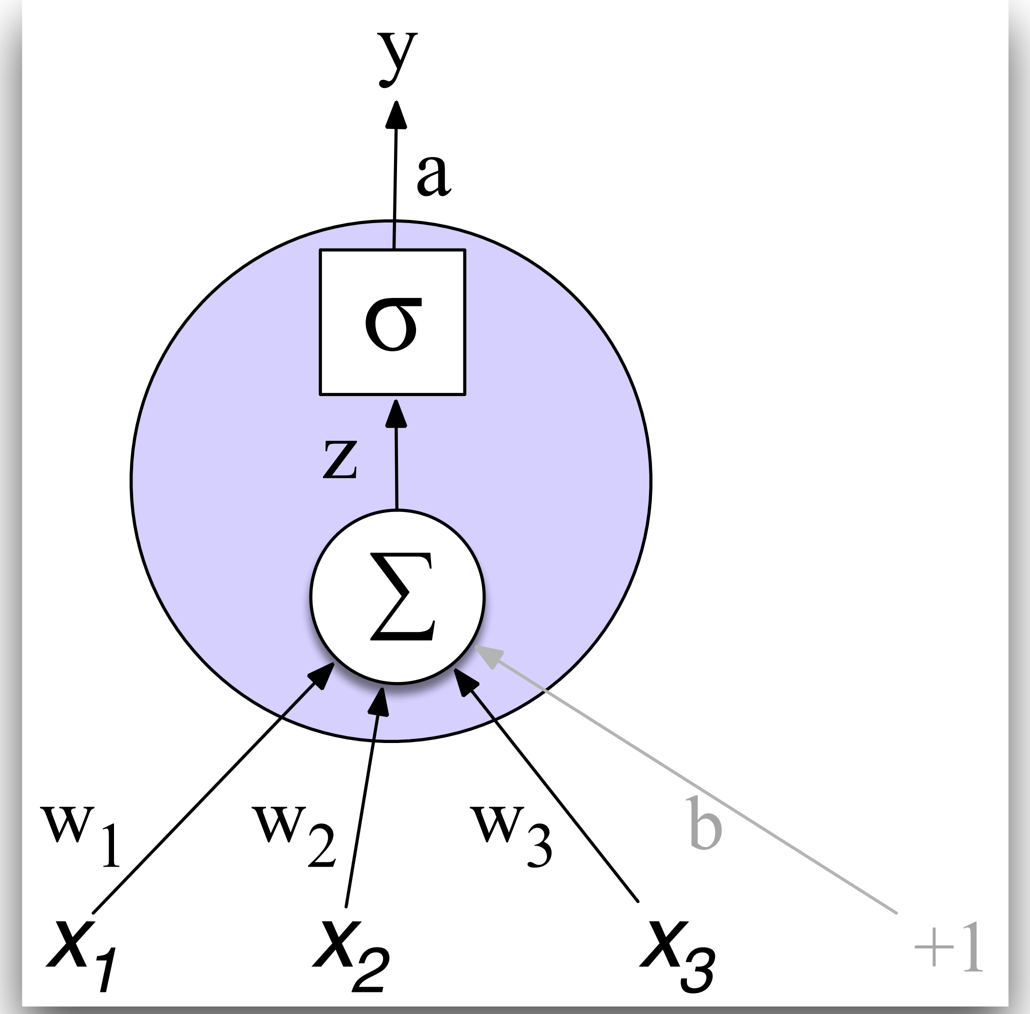
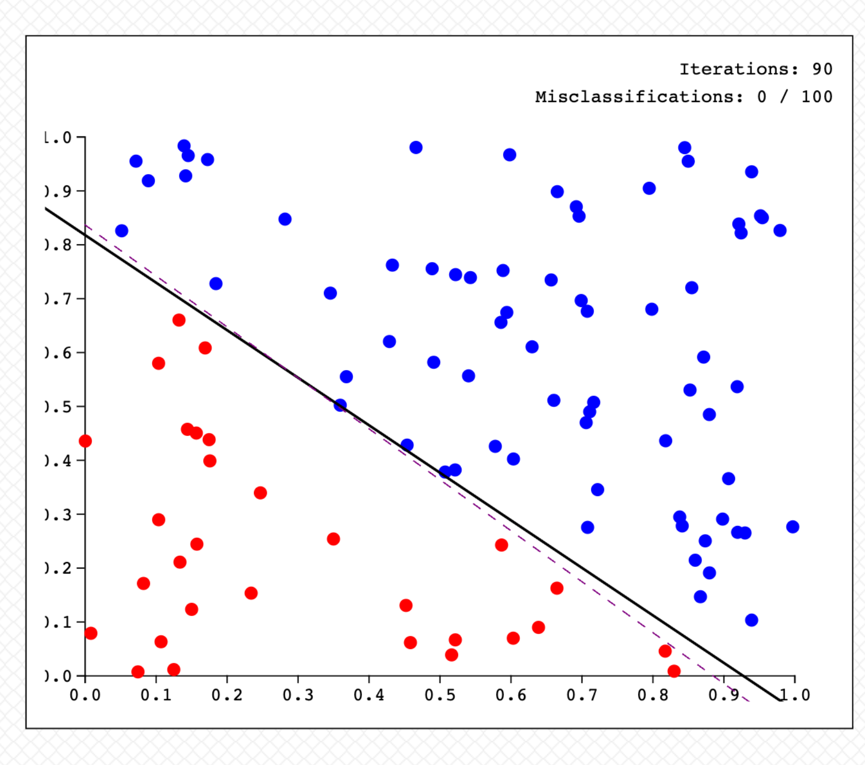
- Perceptron ***iteratively*** **learns a linear decision boundary** for binary classification — we will come back to this intuition later~





The XOR Problem and The First AI Winter

1969 - 1980~ish

How Expressive is a Perceptron?

[Exercise] Let's try representing logical gates!

- Can a Perceptron compute simple functions of input?
- Assume two inputs x_1 and x_2, use the following activation function:

$$y = \begin{cases} 0, & \text{if } \mathbf{w} \cdot \mathbf{x} + b \leq 0 \\ 1, & \text{if } \mathbf{w} \cdot \mathbf{x} + b > 0 \end{cases}$$

AND

x1	x2	y
0	0	0
0	1	0
1	0	0
1	1	1

OR

x1	x2	y
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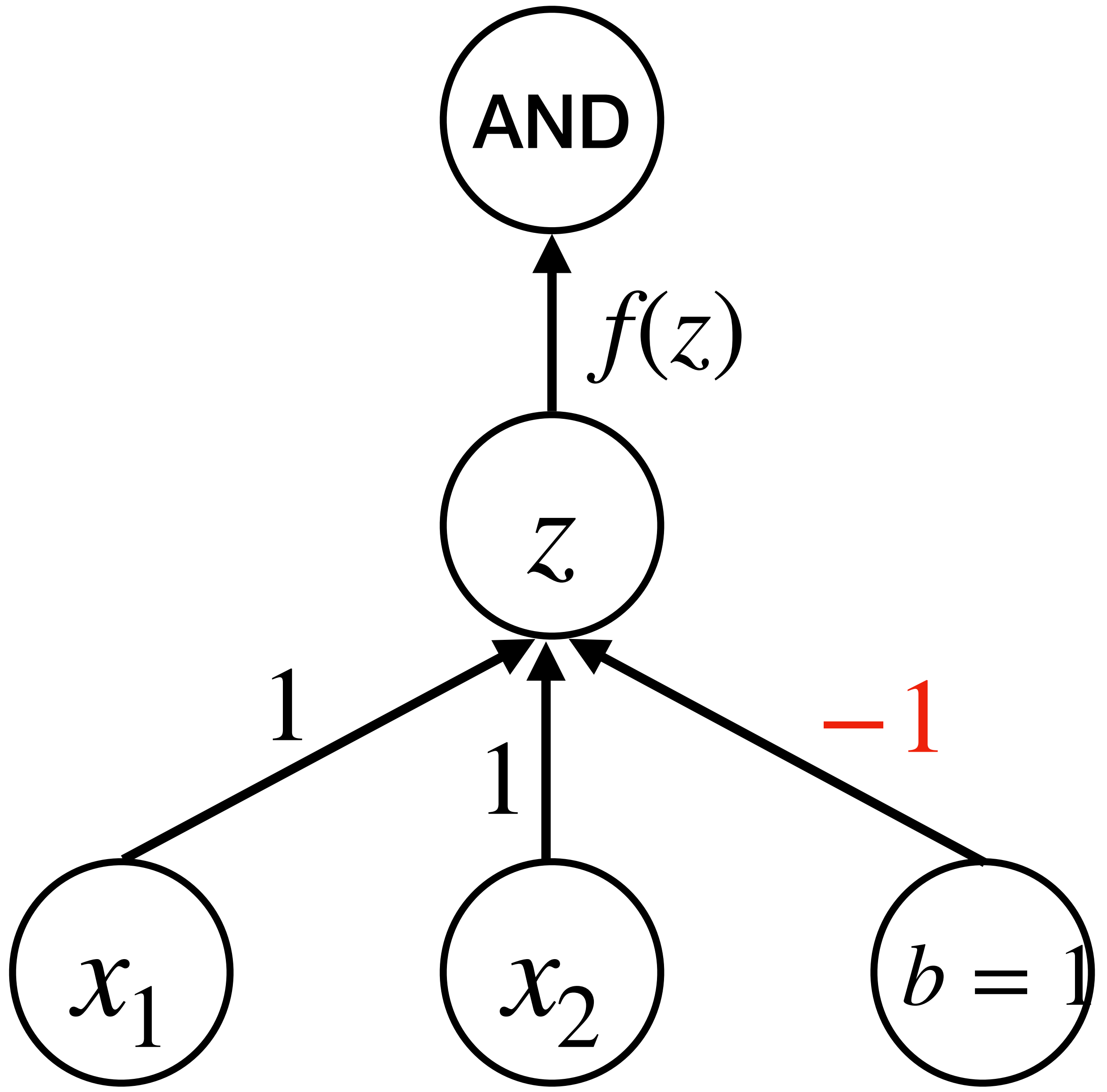

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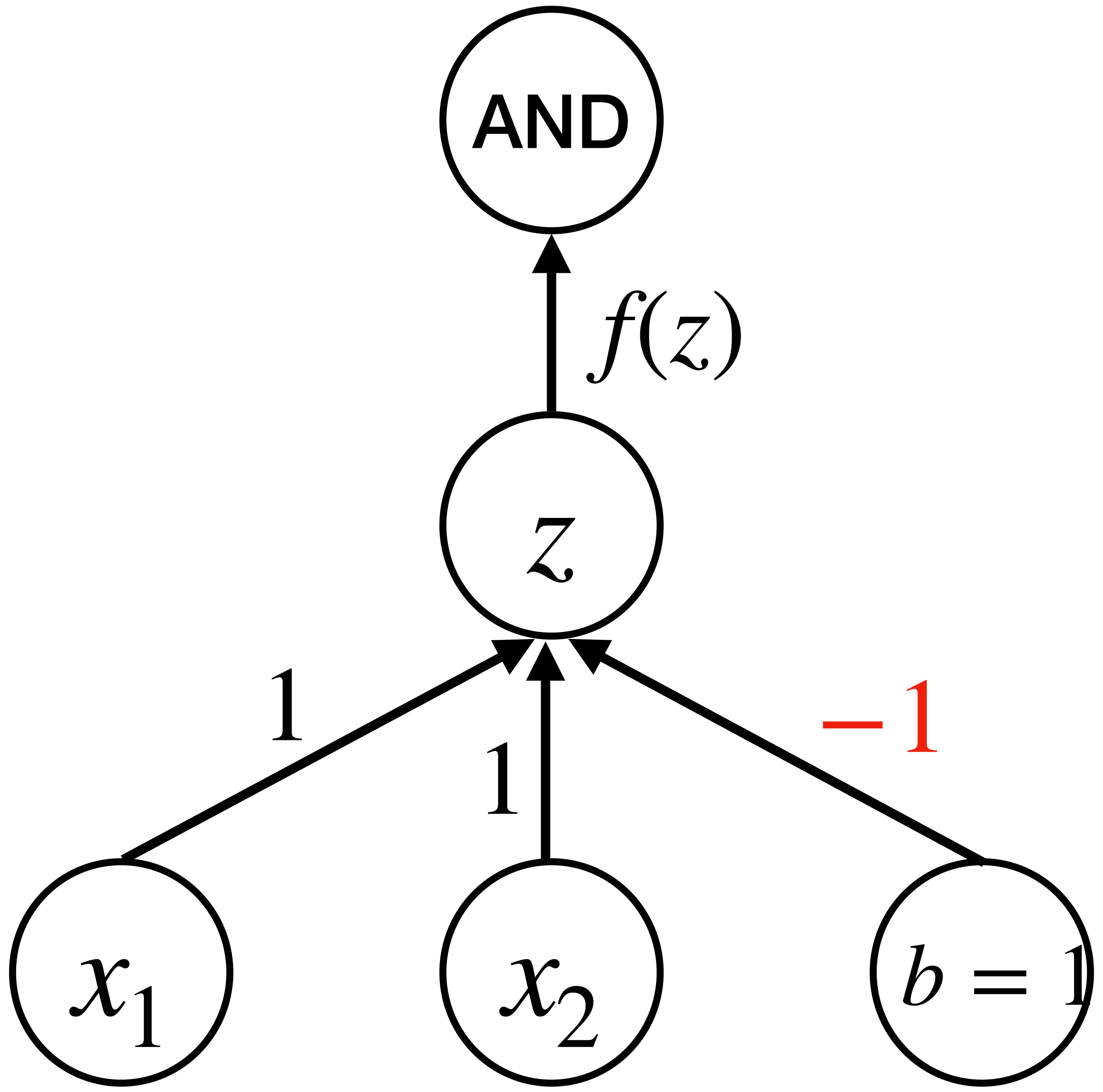

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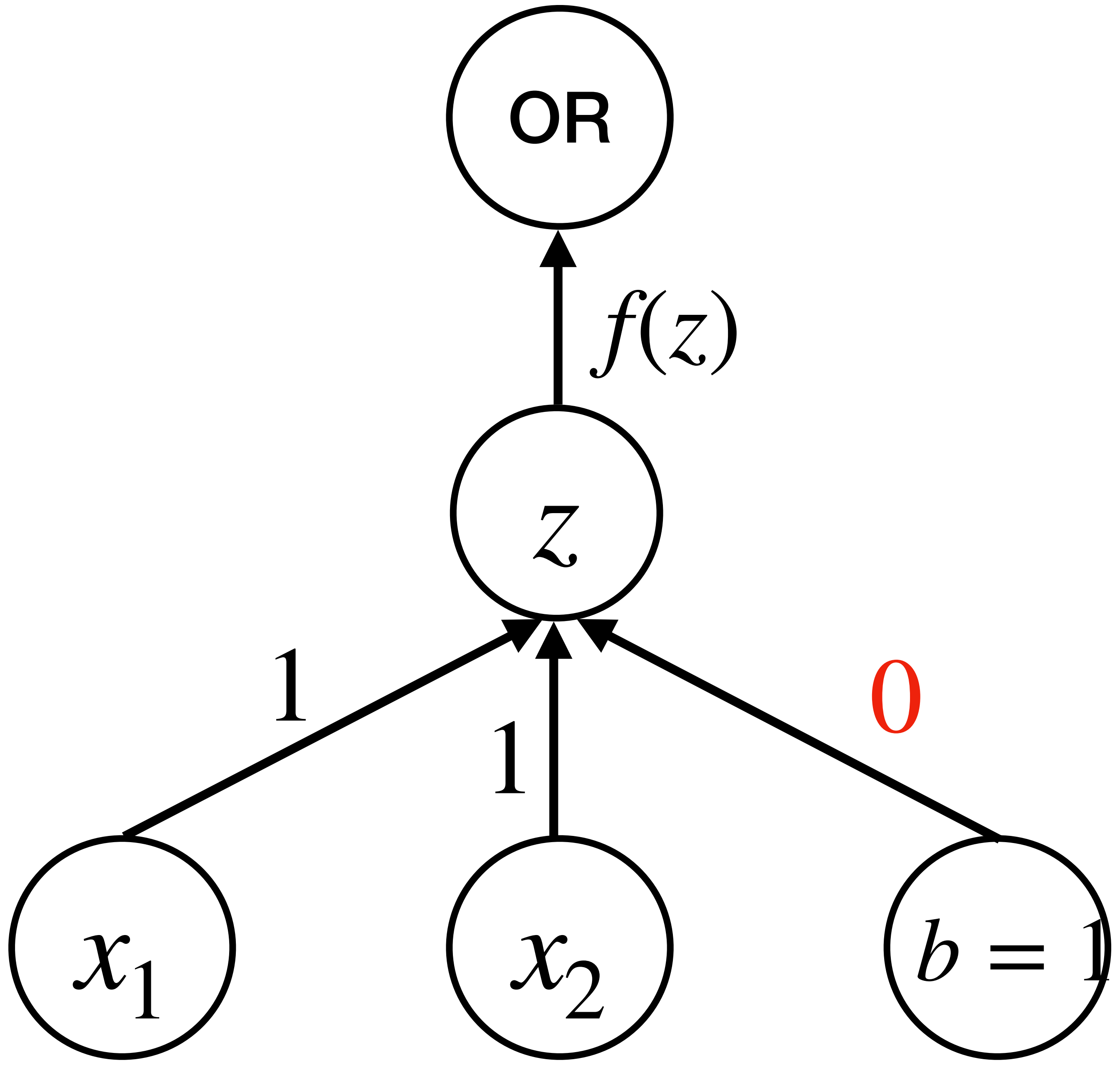

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Introducing another operator: XOR

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  - $w_1x_1 + w_2x_2 + b = 0$
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    - Output = 0 if the input point is on one side of the line;
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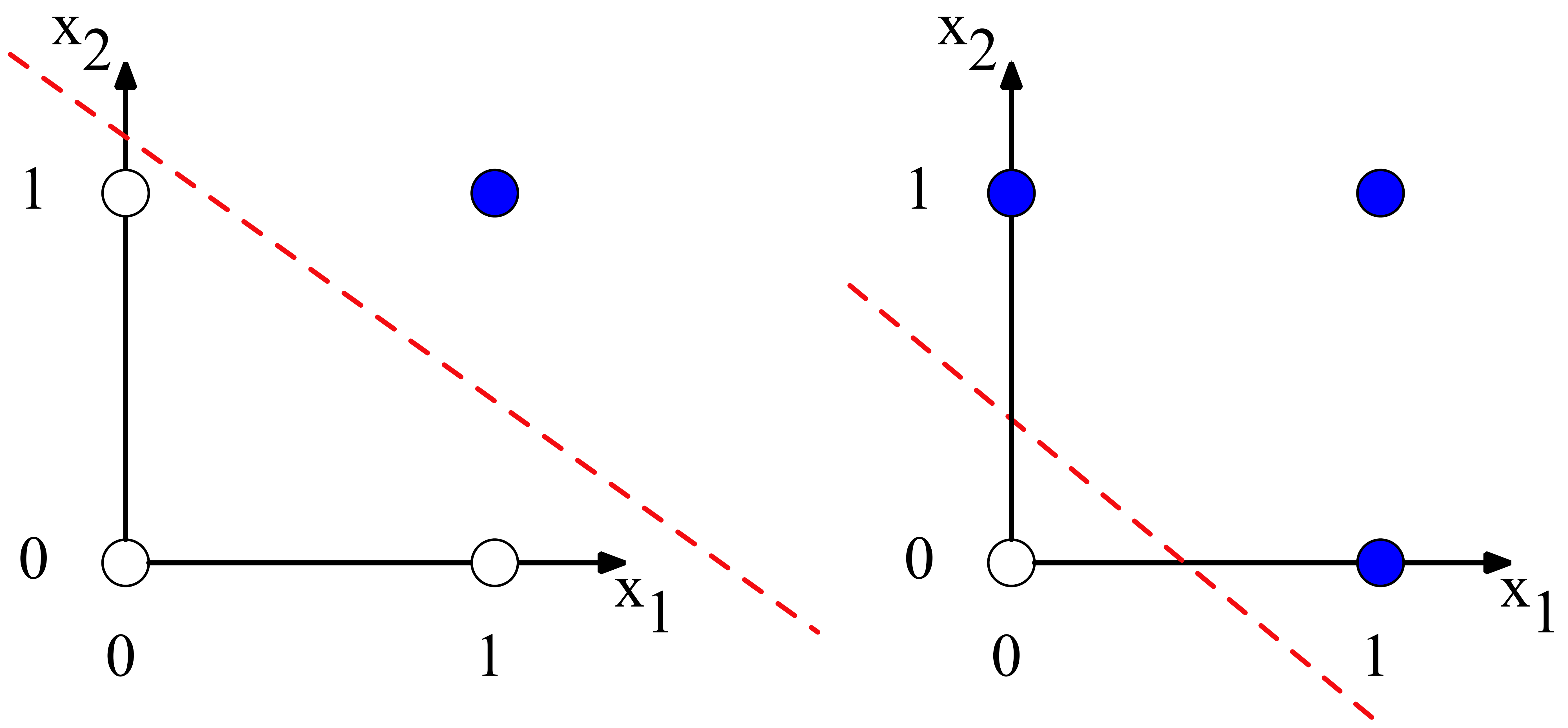
- A good perceptron should be able to find a decision boundary that perfectly separates the 0 points from the 1 points — **linearly separability**!

Perceptron’s Decision Boundaries

Visualizing the decision boundaries for AND and OR



a) x_1 AND x_2

b) x_1 OR x_2

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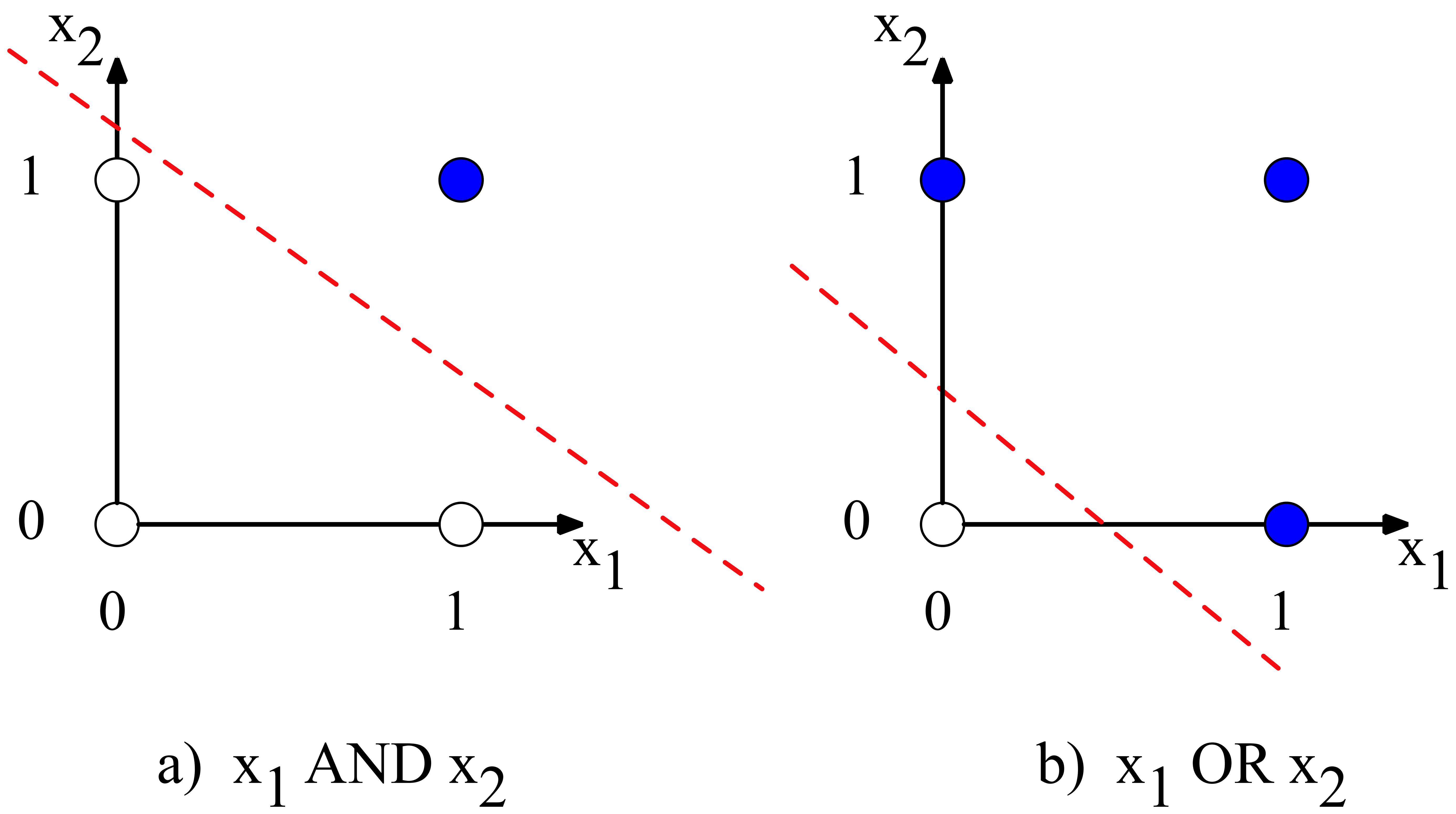

AND and OR are linearly separable!

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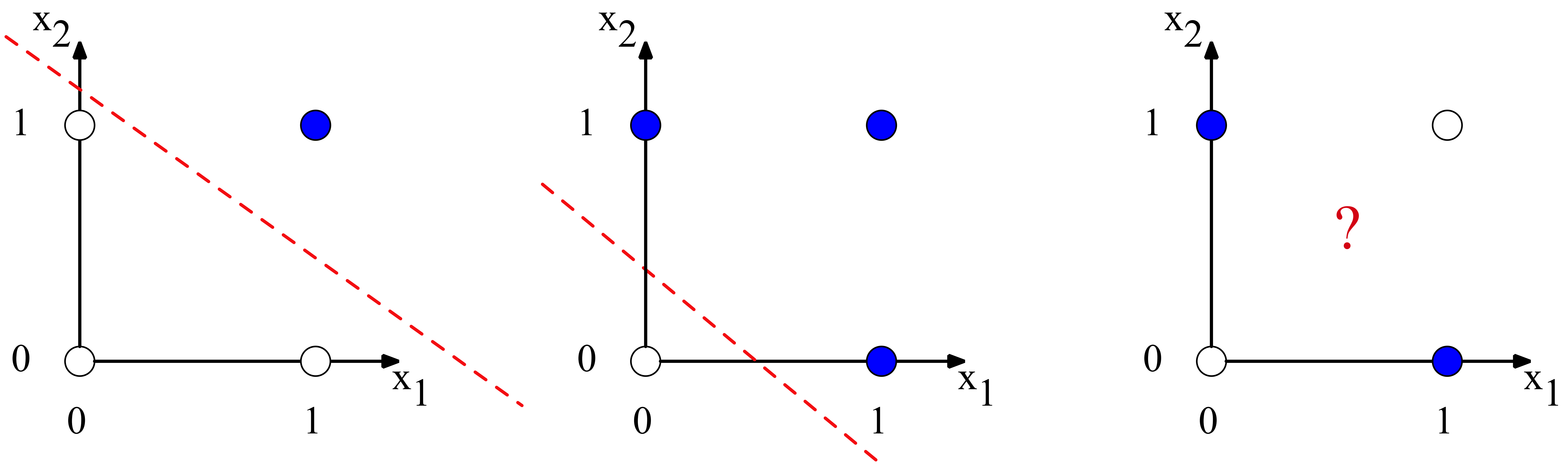

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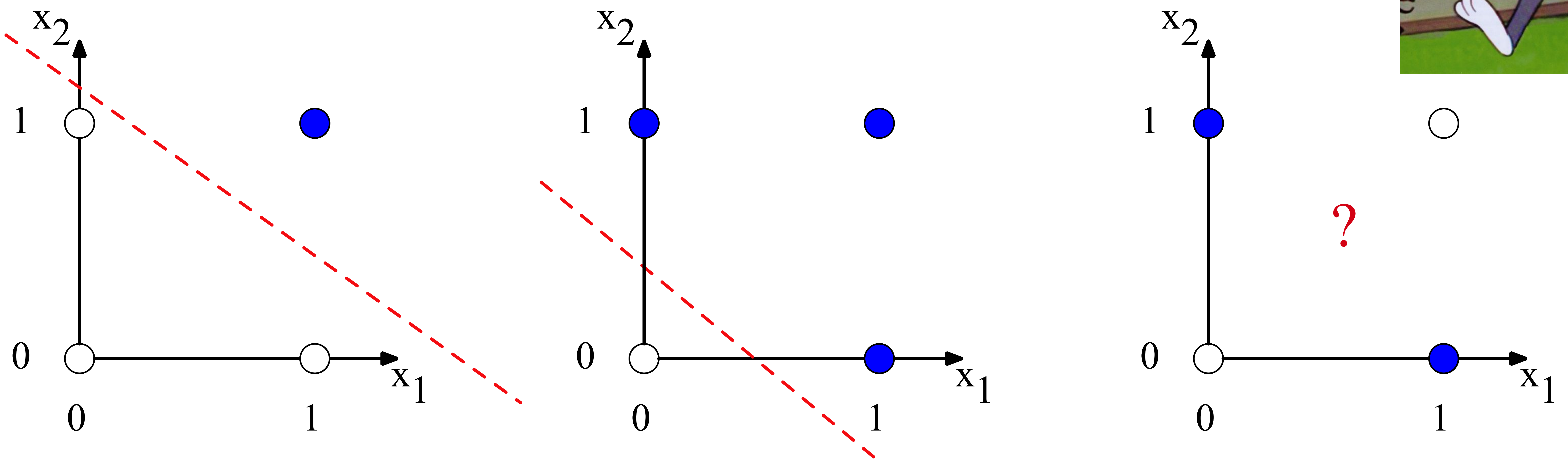

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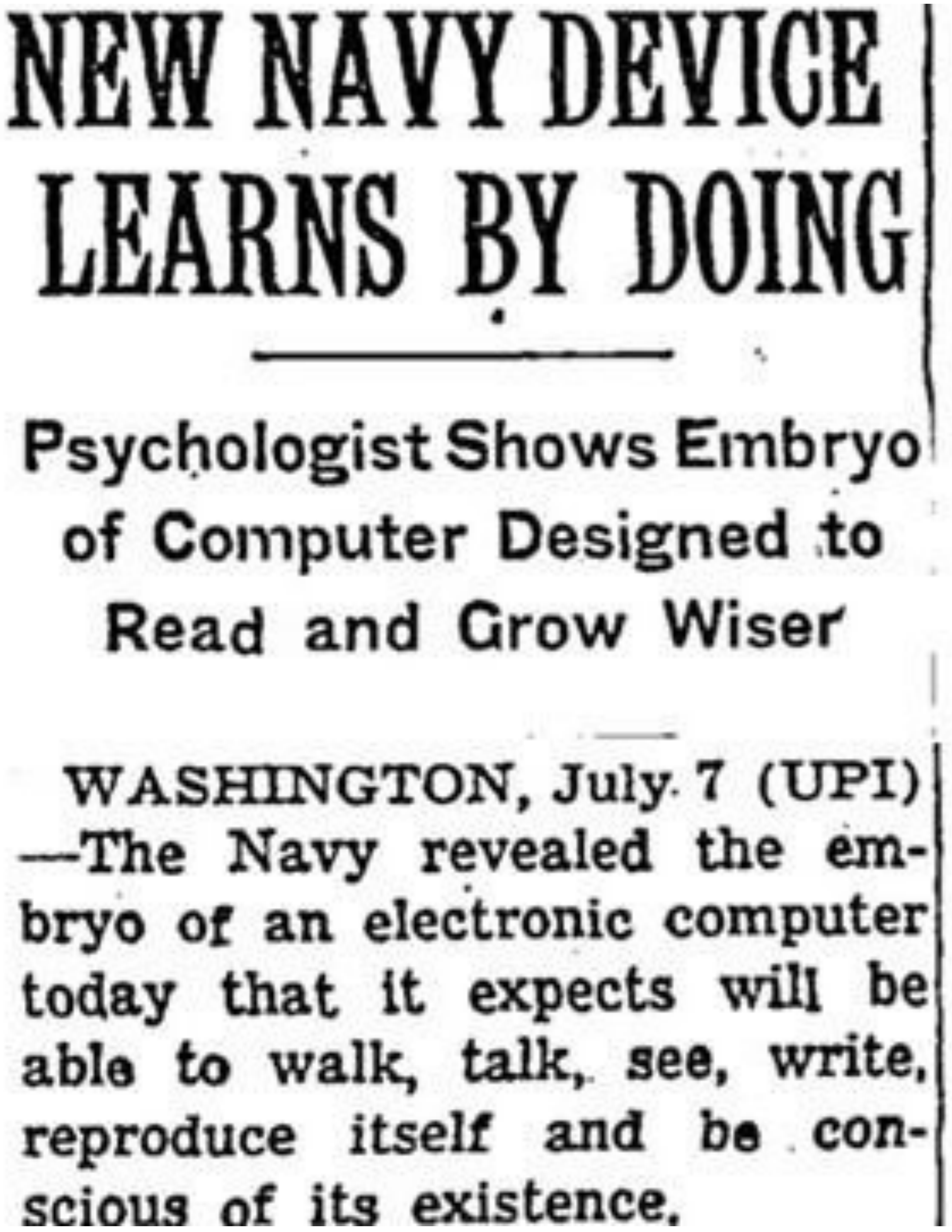
`XOR` is NOT linearly separable!

The AI Winter

10 years from prosperity to fall

- When Perceptron was introduced, it was quite exciting to have a machine that could learn from experience.

- This resulted in some vintage AI hype!
  - ✳ Optimism ran wild: newspapers proclaimed that the perceptron would one day "walk, talk, see, and be conscious of its own existence."
  - ✳ Lots of research fundings.



NEW NAVY DEVICE LEARNS BY DOING

Psychologist Shows Embryo of Computer Designed to Read and Grow Wiser

WASHINGTON, July 7 (UPI) —The Navy revealed the embryo of an electronic computer today that it expects will be able to walk, talk, see, write, reproduce itself and be conscious of its existence.

NYT, via StefanoErmon on Twitter
https://x.com/StefanoErmon/status/936396977218056192?lang=en

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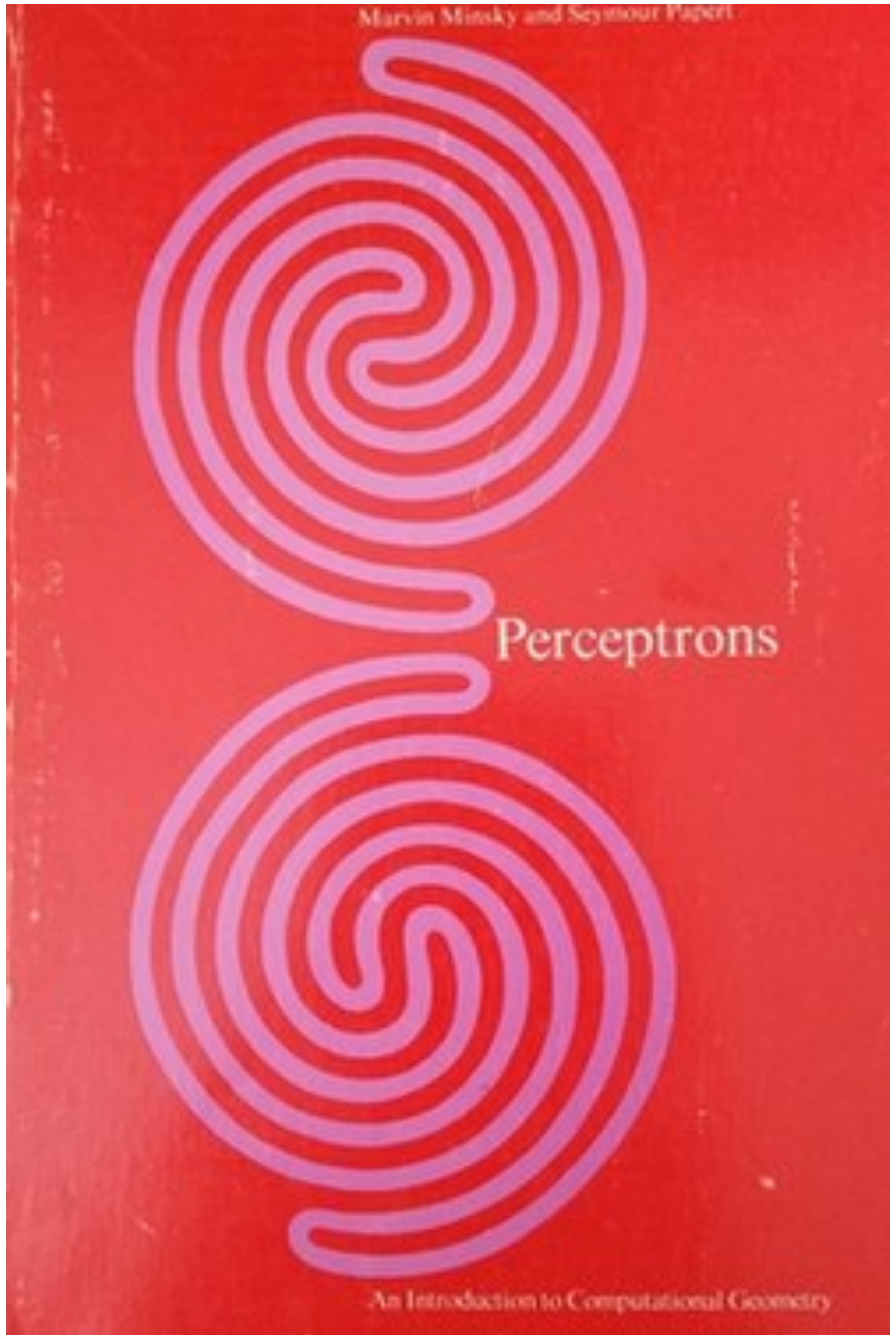

Minsky & Papert 1969

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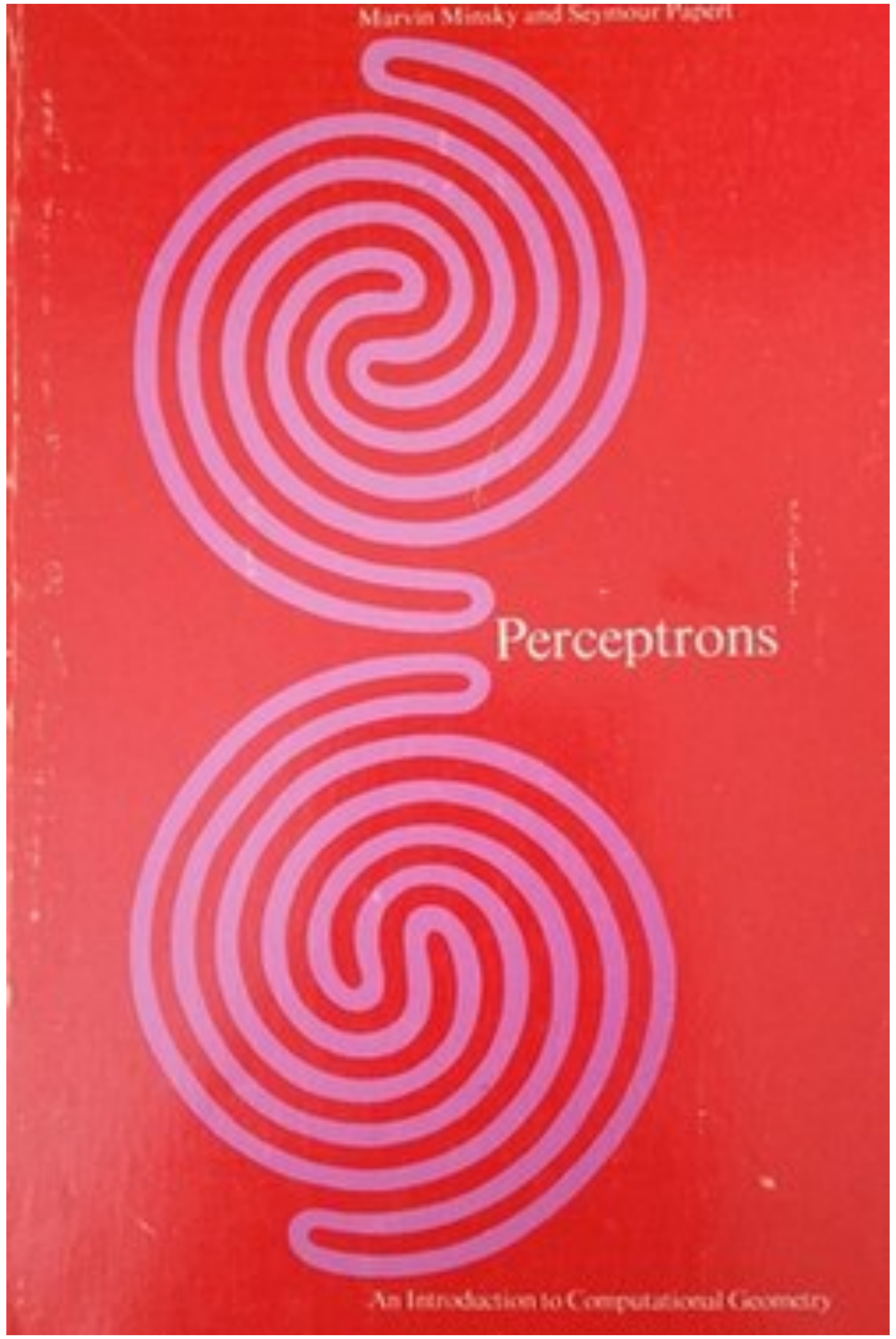
    - Highlighting the XOR problem.



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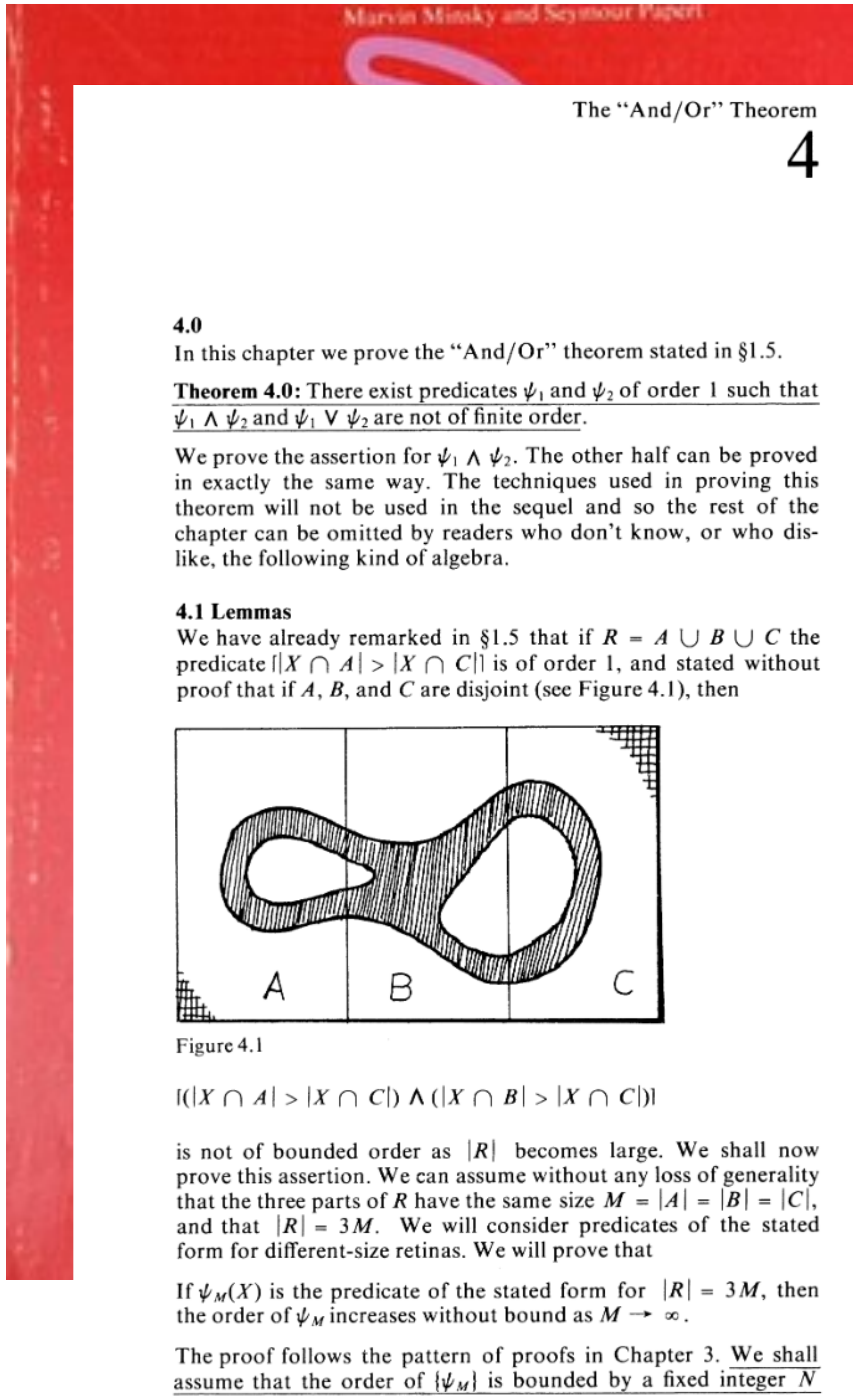

Marvin Minsky and Seymour Papert

The "And/Or" Theorem

4

4.0

In this chapter we prove the "And/Or" theorem stated in §1.5.

Theorem 4.0: There exist predicates ψ_1 and ψ_2 of order 1 such that $\psi_1 \wedge \psi_2$ and $\psi_1 \vee \psi_2$ are not of finite order.

We prove the assertion for $\psi_1 \wedge \psi_2$. The other half can be proved in exactly the same way. The techniques used in proving this theorem will not be used in the sequel and so the rest of the chapter can be omitted by readers who don't know, or who dislike, the following kind of algebra.

4.1 Lemmas

We have already remarked in §1.5 that if $R = A \cup B \cup C$ the predicate $\lceil |X \cap A| > |X \cap C| \rceil$ is of order 1, and stated without proof that if A, B, and C are disjoint (see Figure 4.1), then

A B C

Figure 4.1

$$\lceil (|X \cap A| > |X \cap C|) \wedge (|X \cap B| > |X \cap C|) \rceil$$

is not of bounded order as $|R|$ becomes large. We shall now prove this assertion. We can assume without any loss of generality that the three parts of R have the same size $M = |A| = |B| = |C|$, and that $|R| = 3M$. We will consider predicates of the stated form for different-size retinas. We will prove that

If $\psi_M(X)$ is the predicate of the stated form for $|R| = 3M$, then the order of ψ_M increases without bound as $M \to \infty$.

The proof follows the pattern of proofs in Chapter 3. We shall assume that the order of $\{\psi_M\}$ is bounded by a fixed integer N

Minsky & Papert 1969

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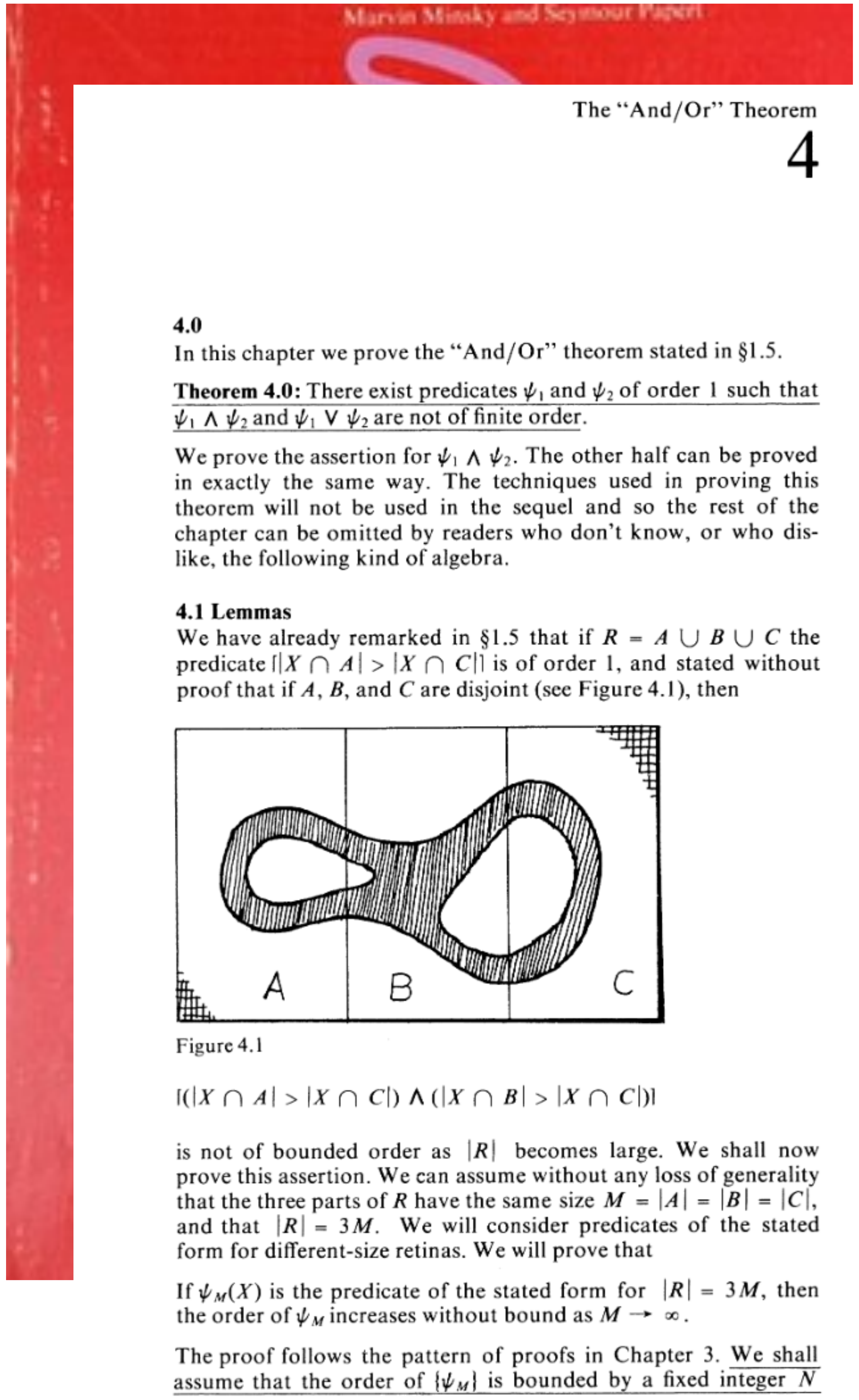
  - Highlighting the XOR problem.
- This resulted in the first AI winter:
  - ✳ Research funding agencies concluded that neural networks had hit a theoretical dead end.
  - ✳ AI research shifted towards symbolic logic, expert systems, rule-based reasoning (n-gram models were born in this stage!)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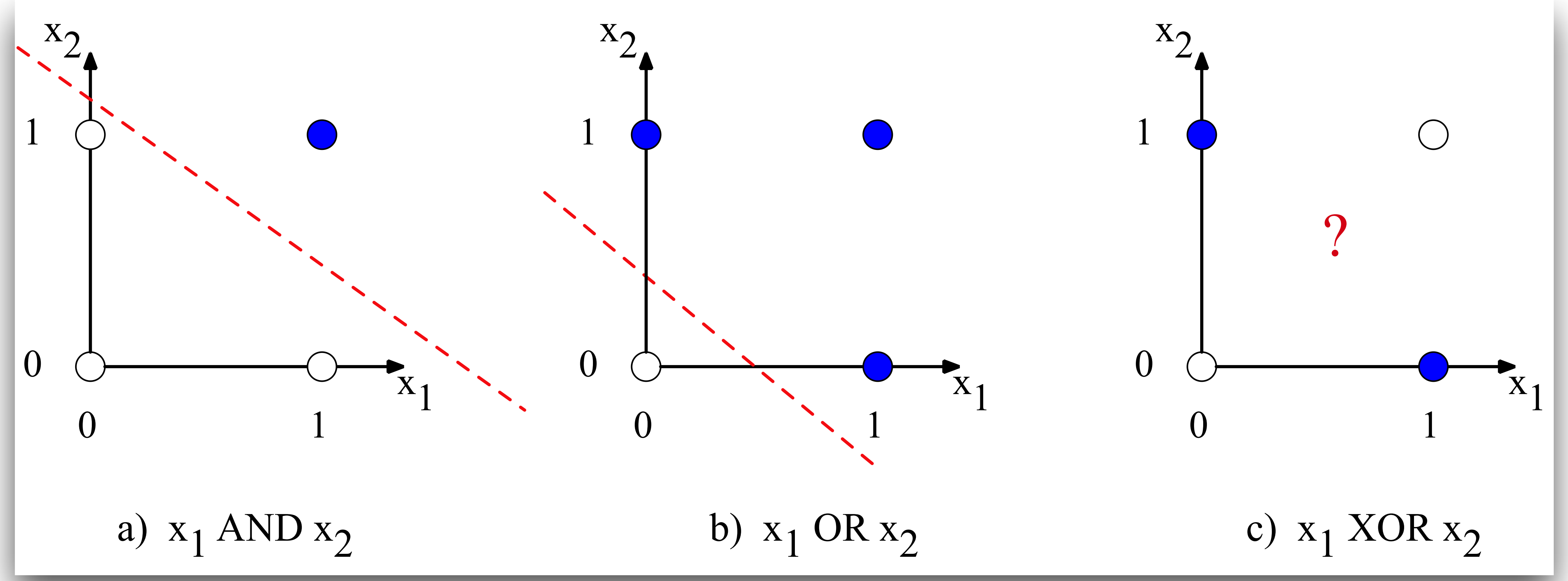
Interium Summary 2

- **The XOR Problem** demonstrates a crucial limit of a single-neuron perceptron.
- Perceptron can only learn linearly separable patterns.
- This triggers **the first AI winter** after the initial AI hype for 10 years.



The Revival of Neural Networks

1980s

One Solution to the XOR Problem

What if we use more than one layer?

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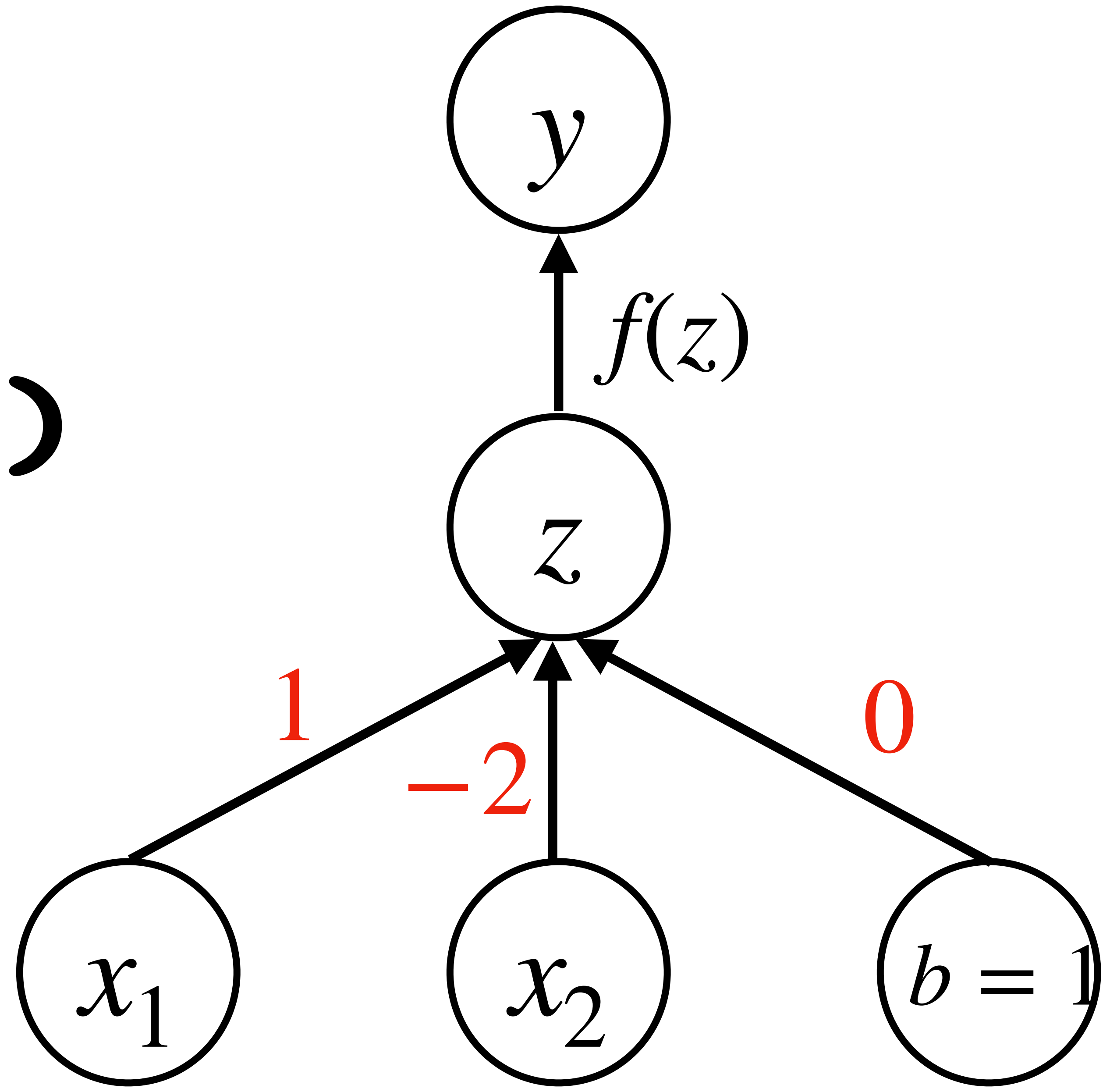

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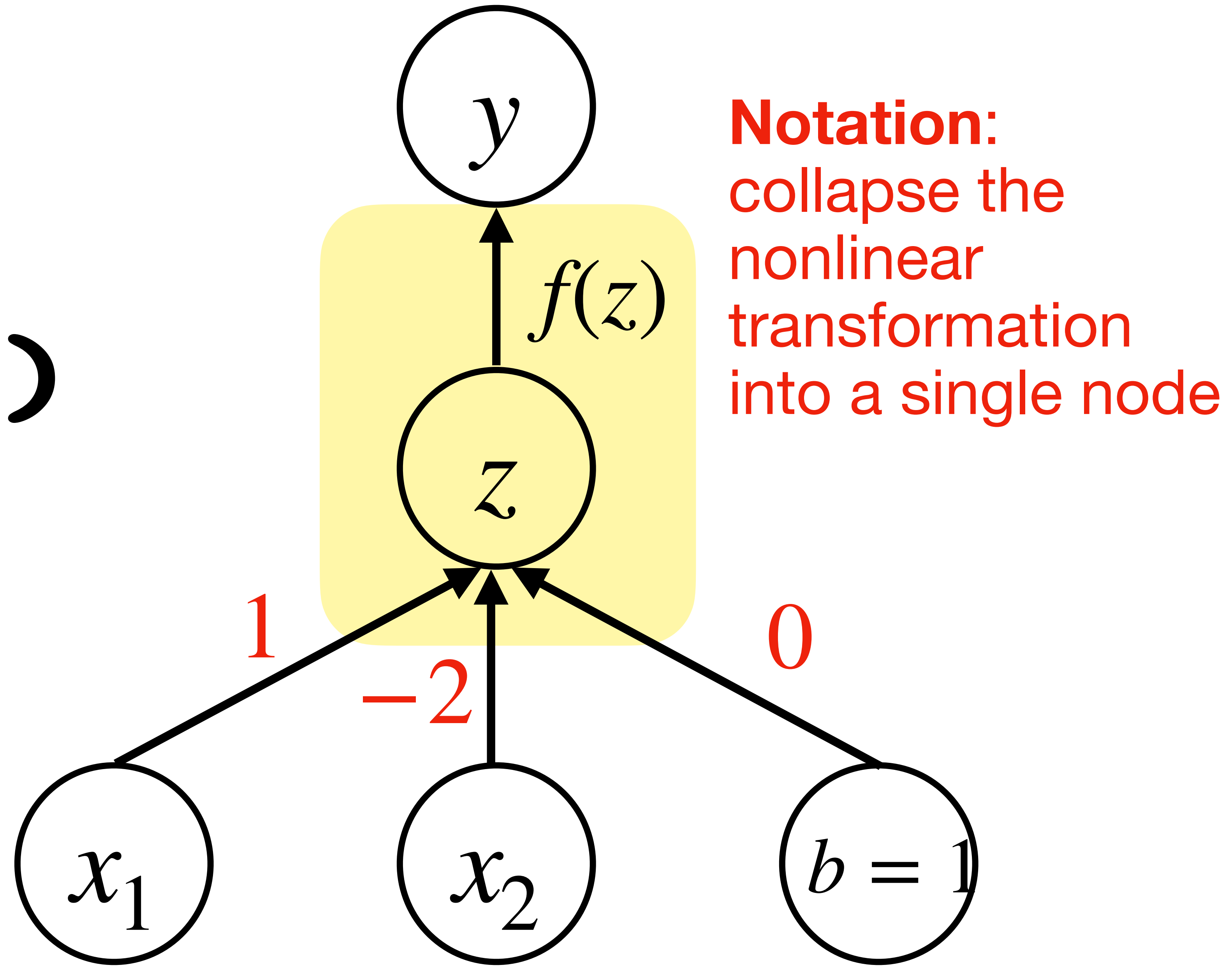

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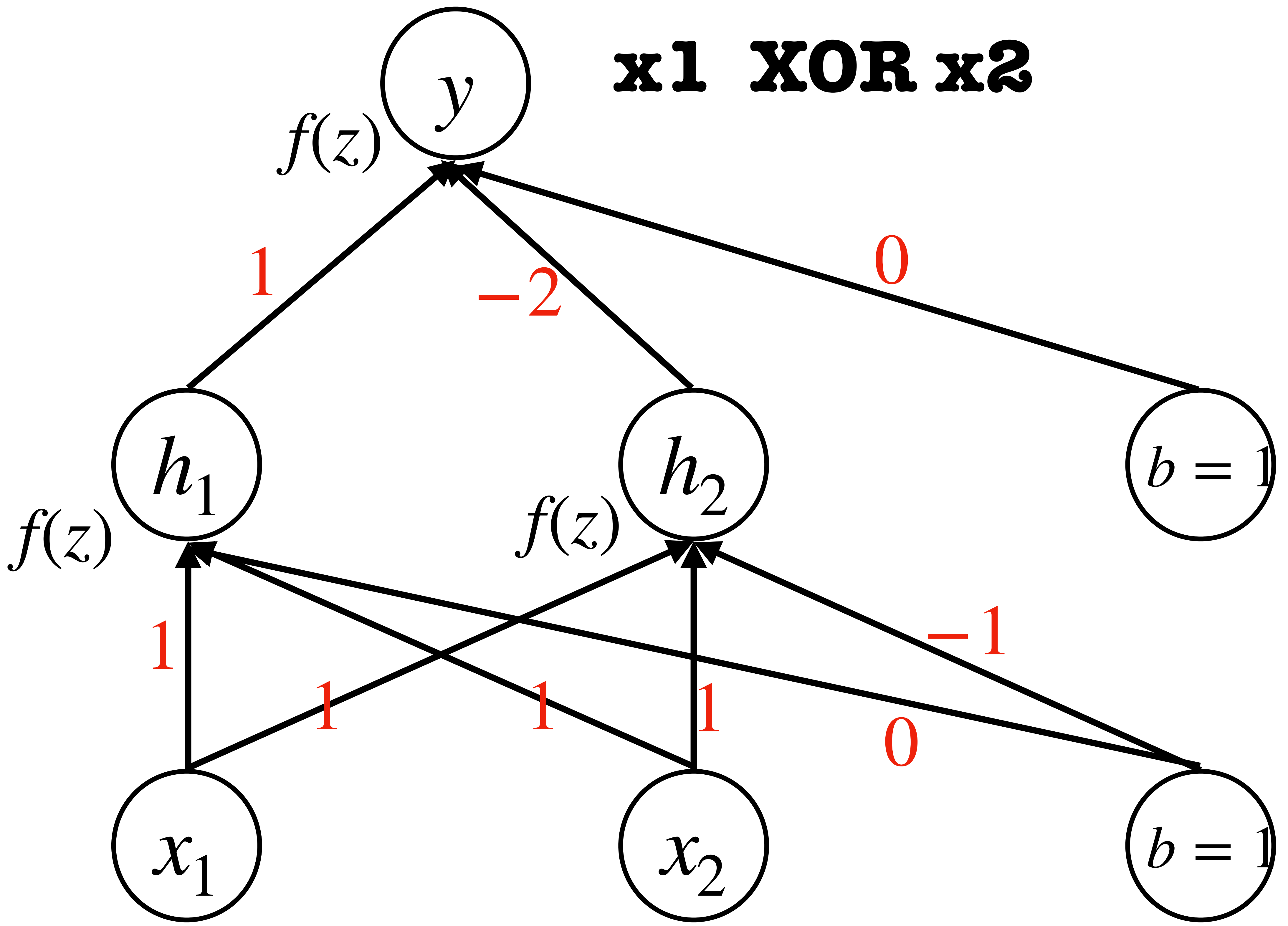

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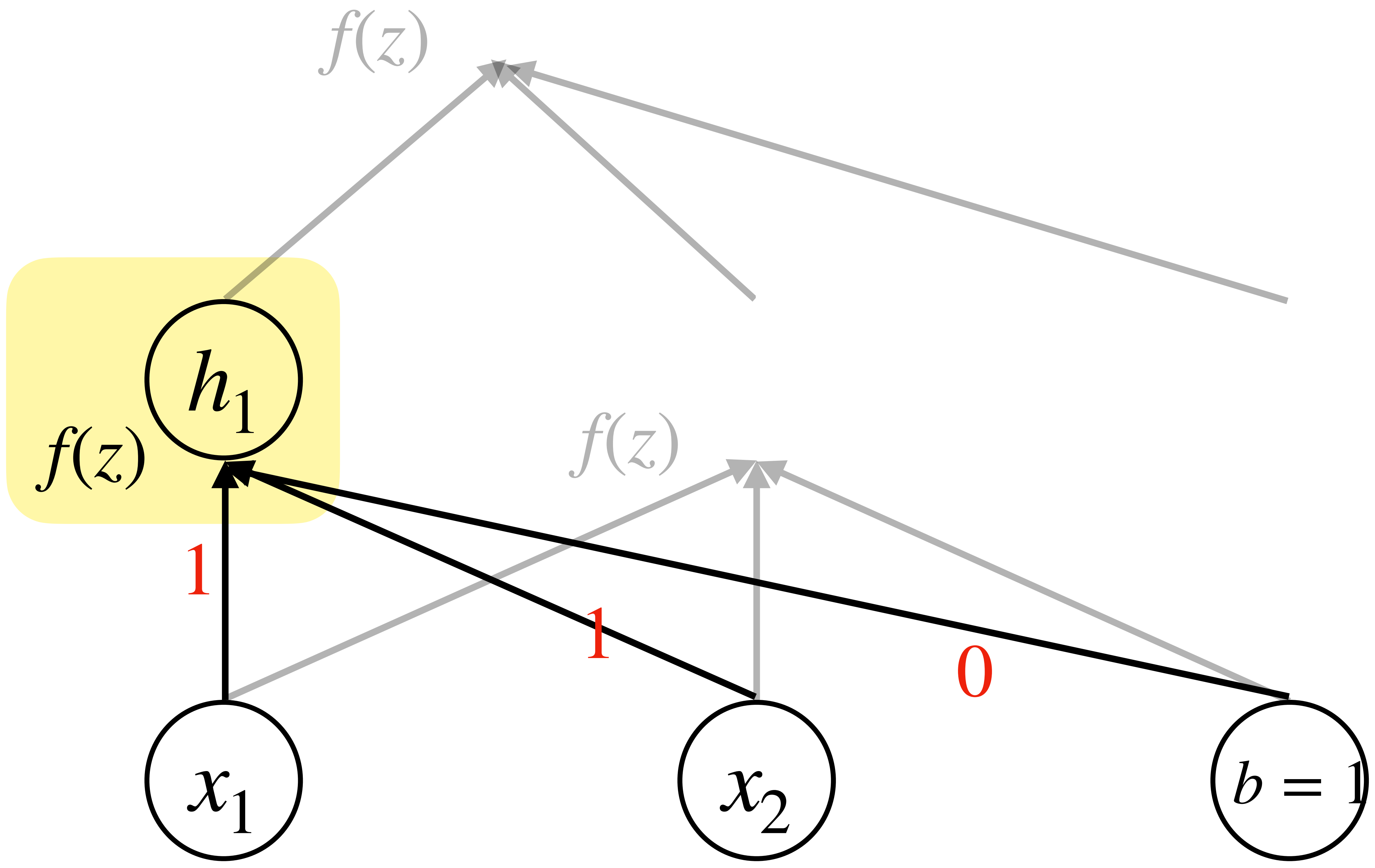
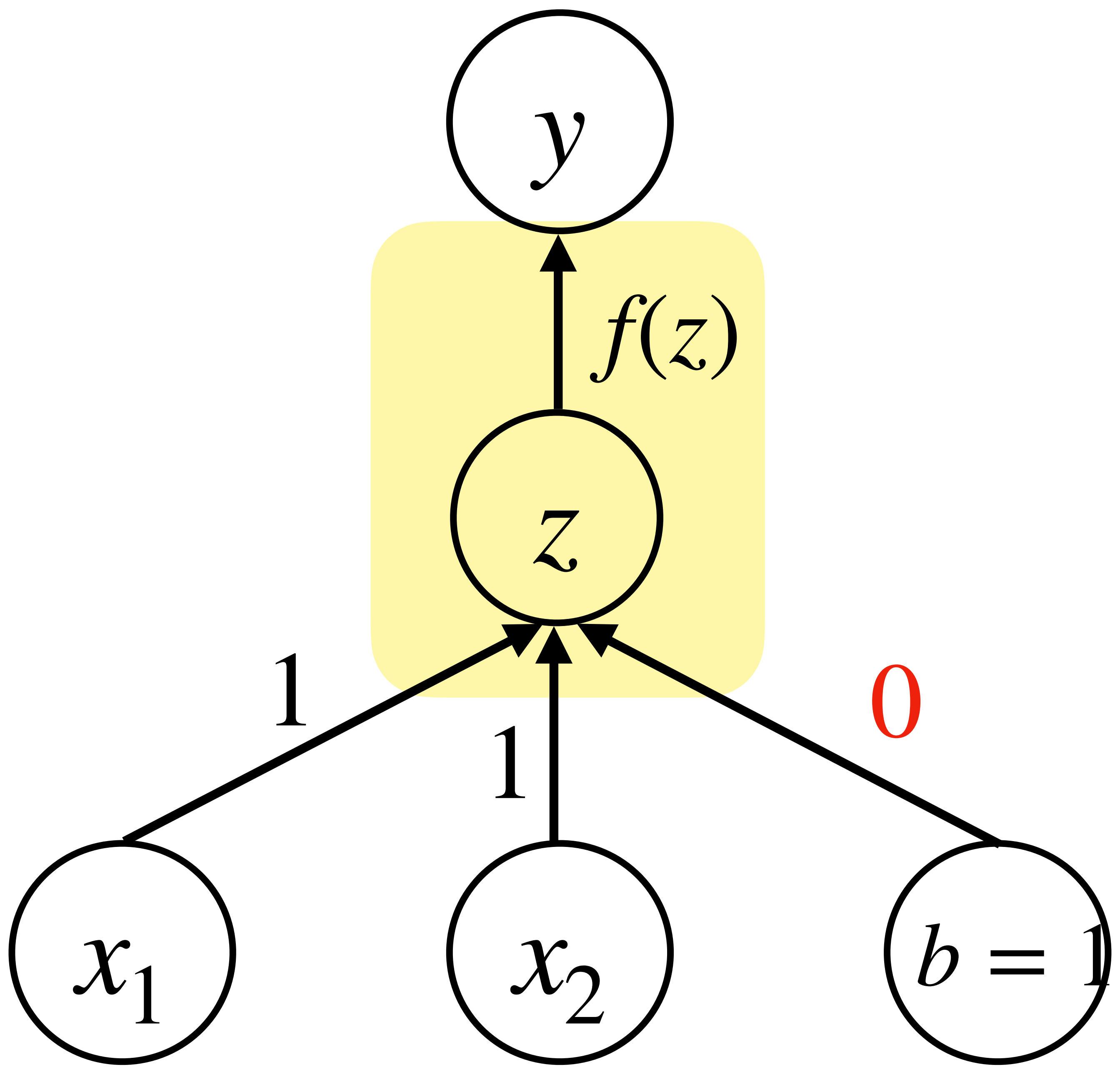
- Component 1: the **OR** gate



x1 OR x2

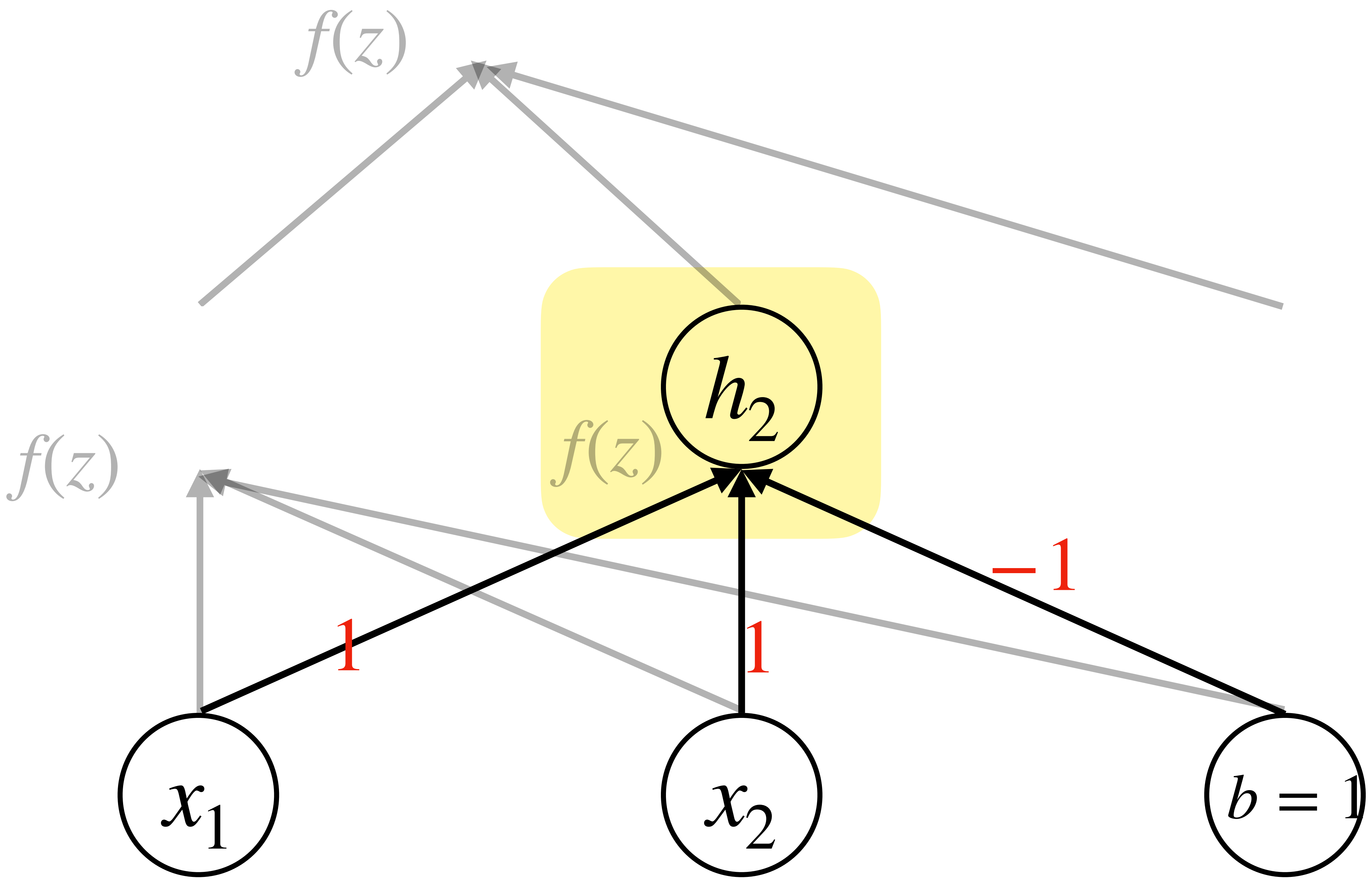


$$y = \begin{cases} 0, & \text{if } \mathbf{w} \cdot \mathbf{x} + b \le 0 \\ 1, & \text{if } \mathbf{w} \cdot \mathbf{x} + b > 0 \end{cases}$$

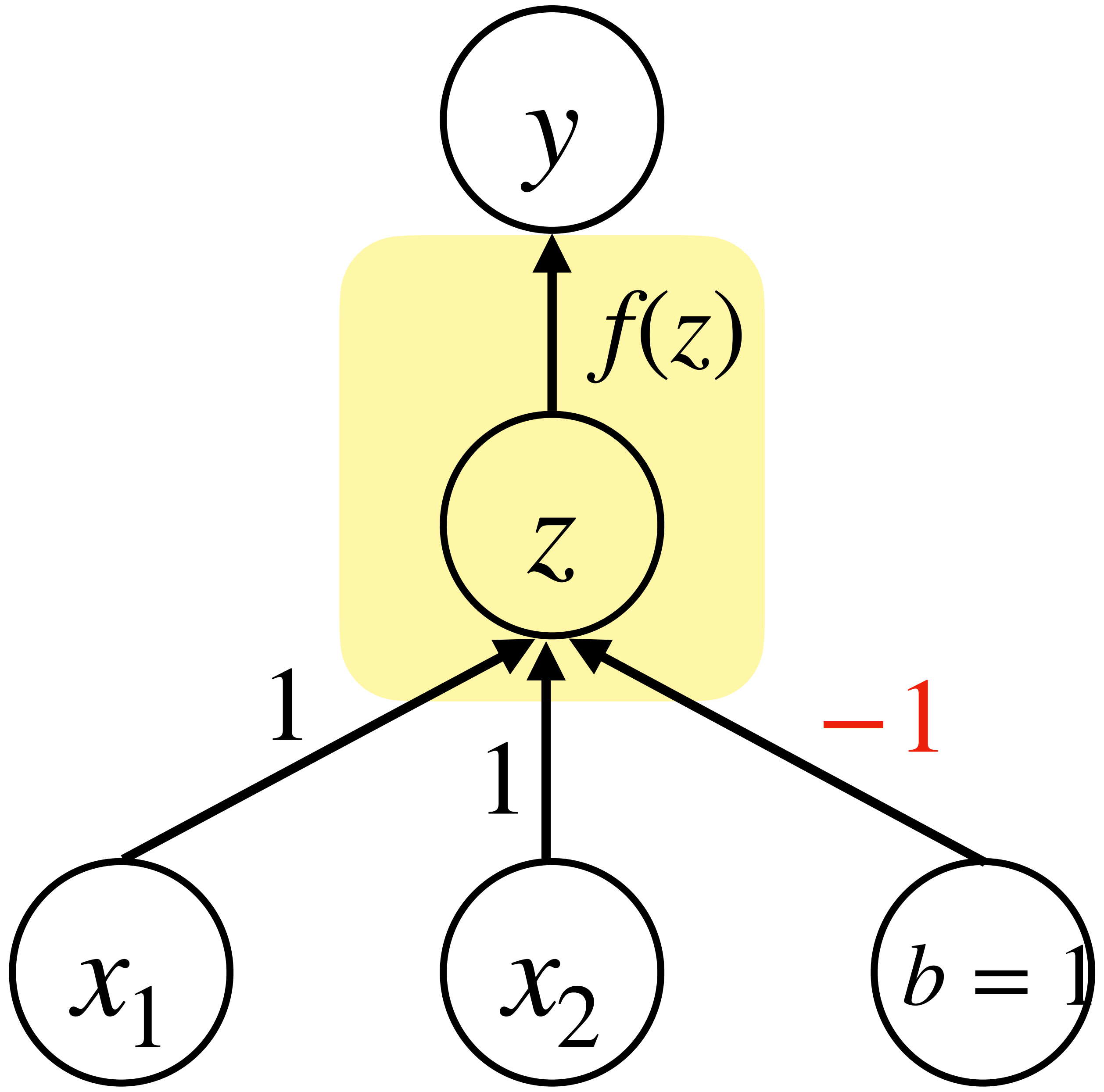
One Solution to the XOR Problem

What if we use more than one layers?

- Component 2: the **AND** gate



💡 **x1 AND x2**



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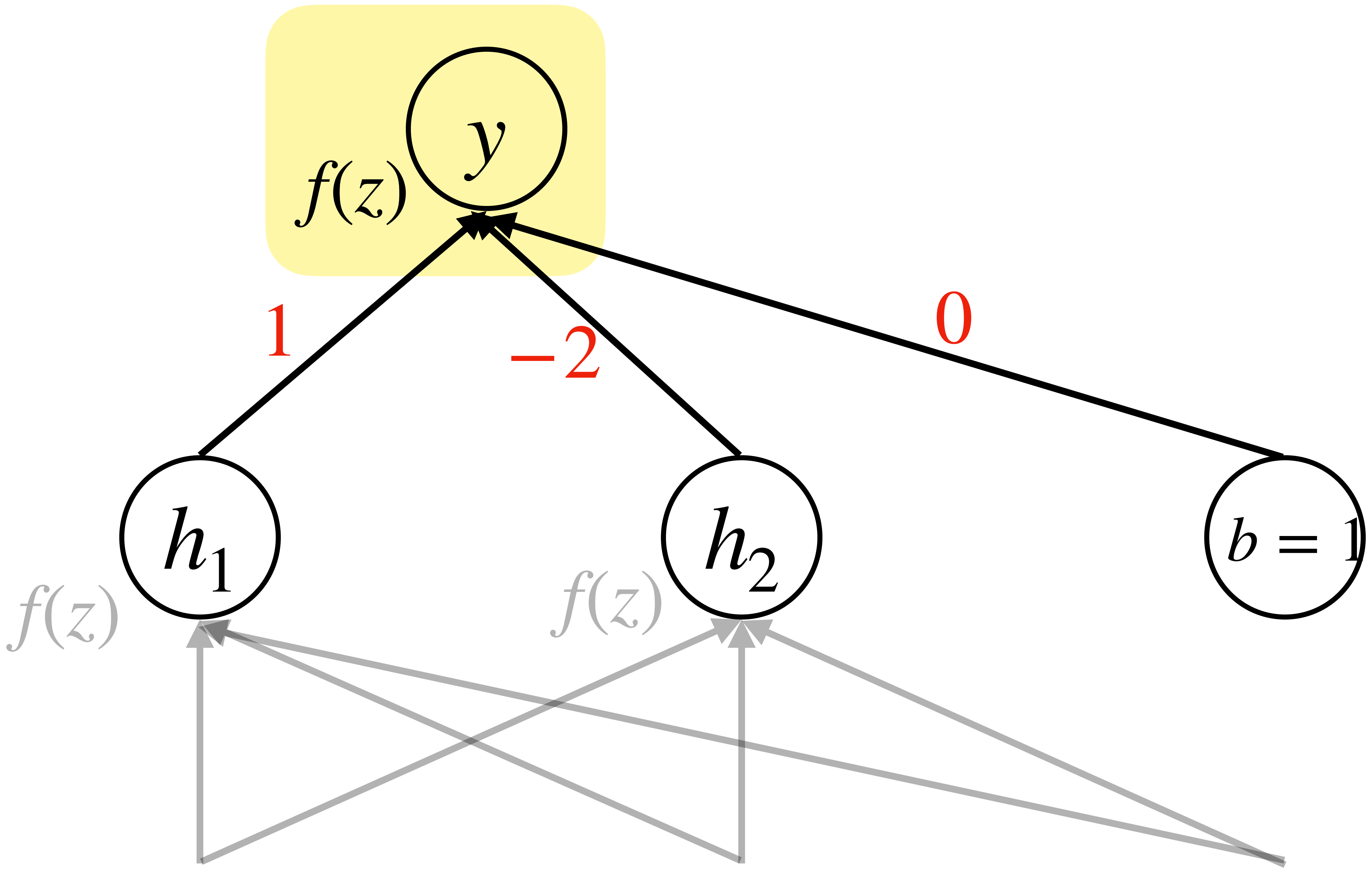
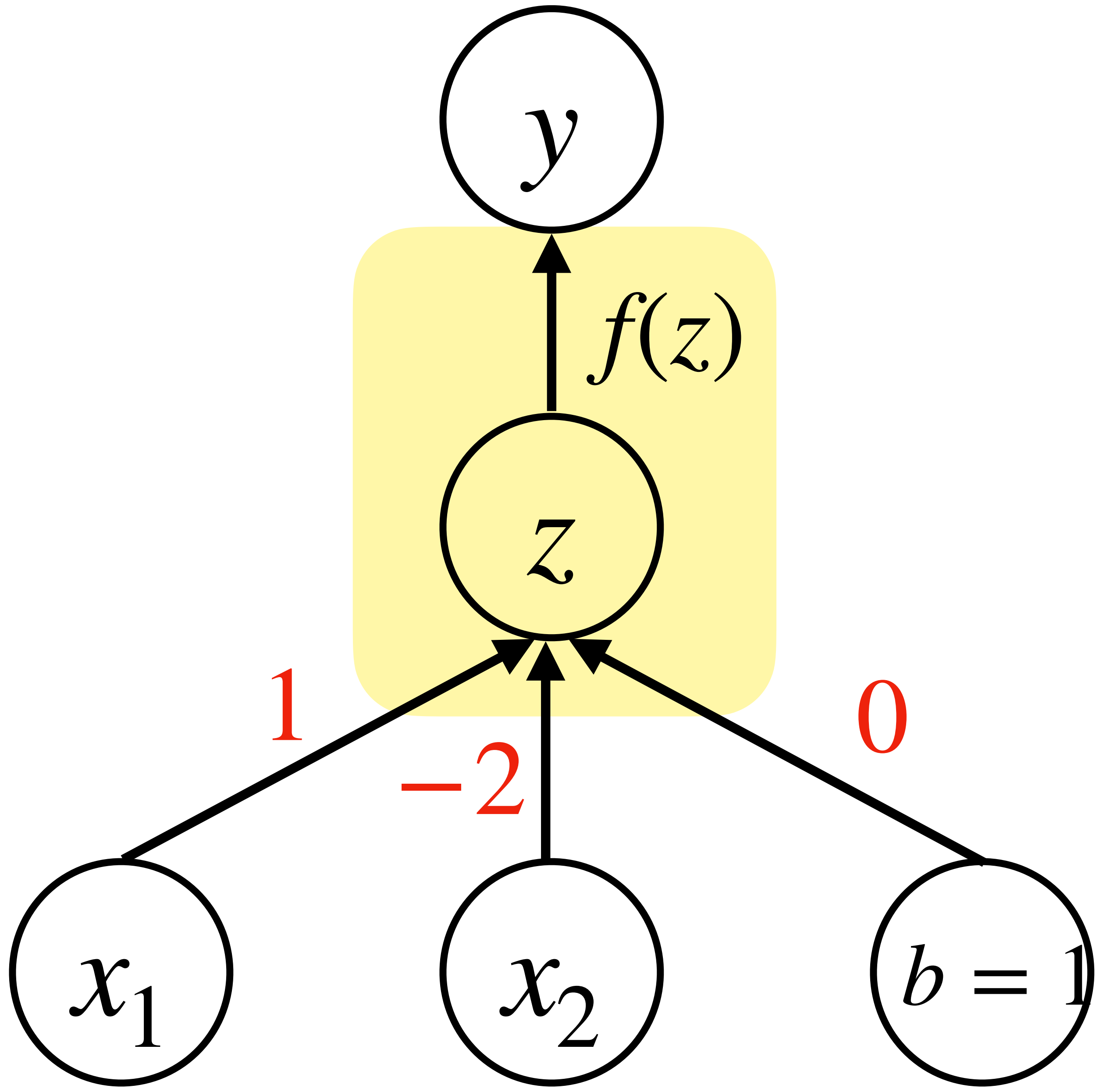
One Solution to the XOR Problem

What if we use more than one layers?

- Component 3: the **AND** gate with **NOT**

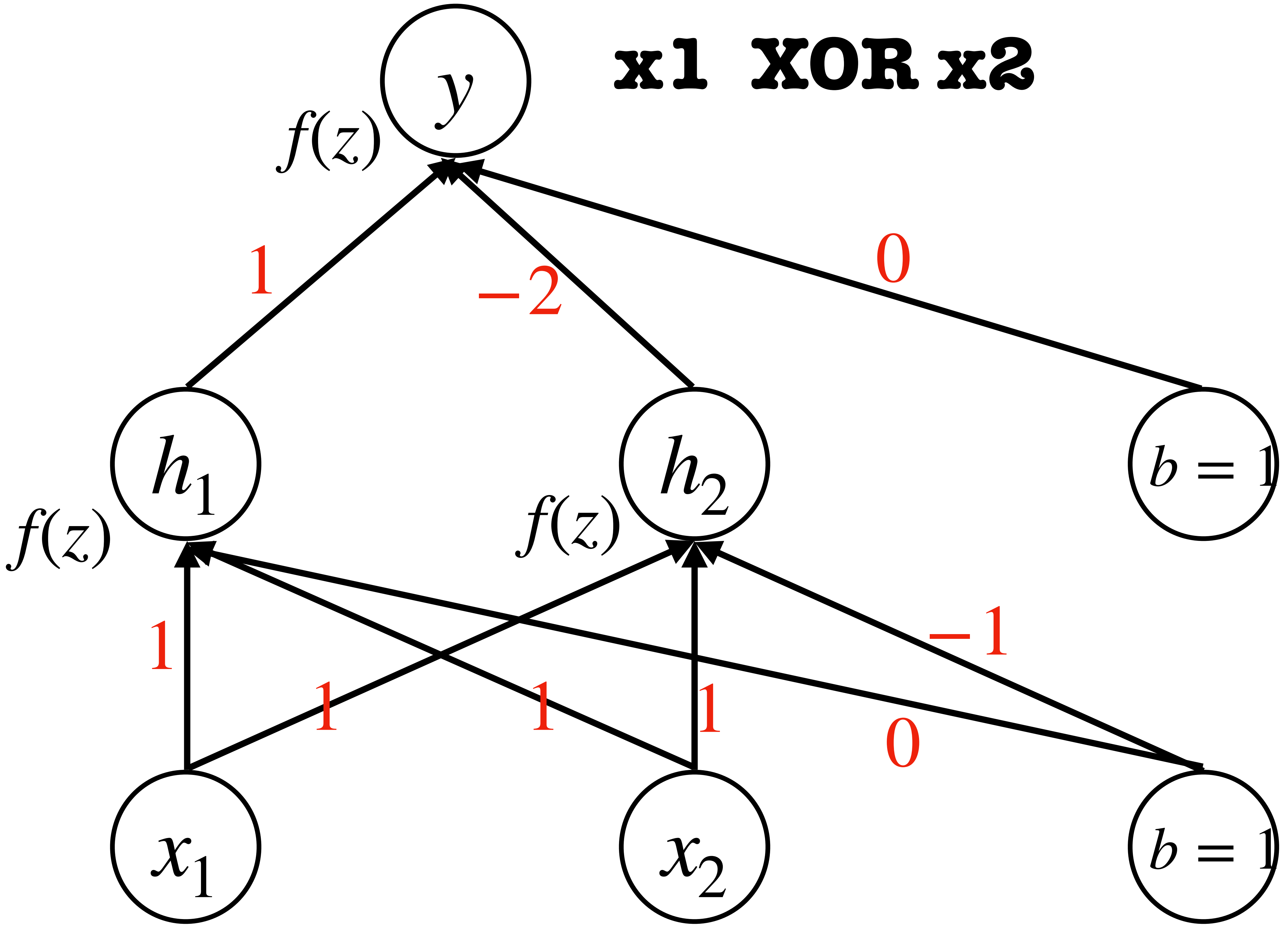


x1 AND (NOT x2)



One Solution to the XOR Problem

Let's verify!



after transformation through f

x1	x2	$h1_0$	$h2_0$	h1	h2	y_0	y
0	0	0	-1	0	0	0	0
0	1	1	0	1	0	1	1
1	0	1	0	1	0	1	1
1	1	2	1	1	1	-1	0

Before transformation through f

One Solution to the XOR Problem

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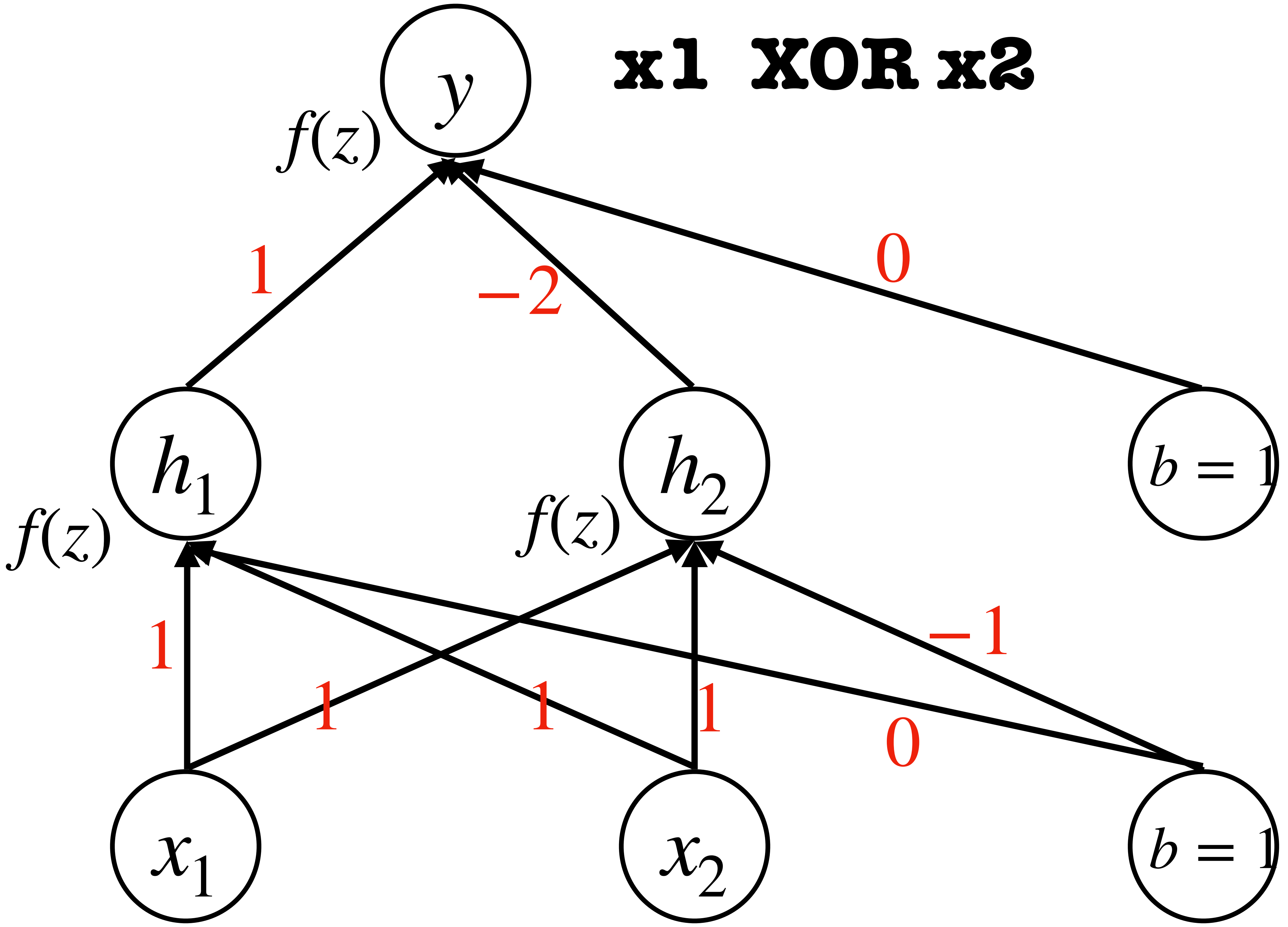

x1	x2	$h1_0$	$h2_0$	h1	h2	y_0	y
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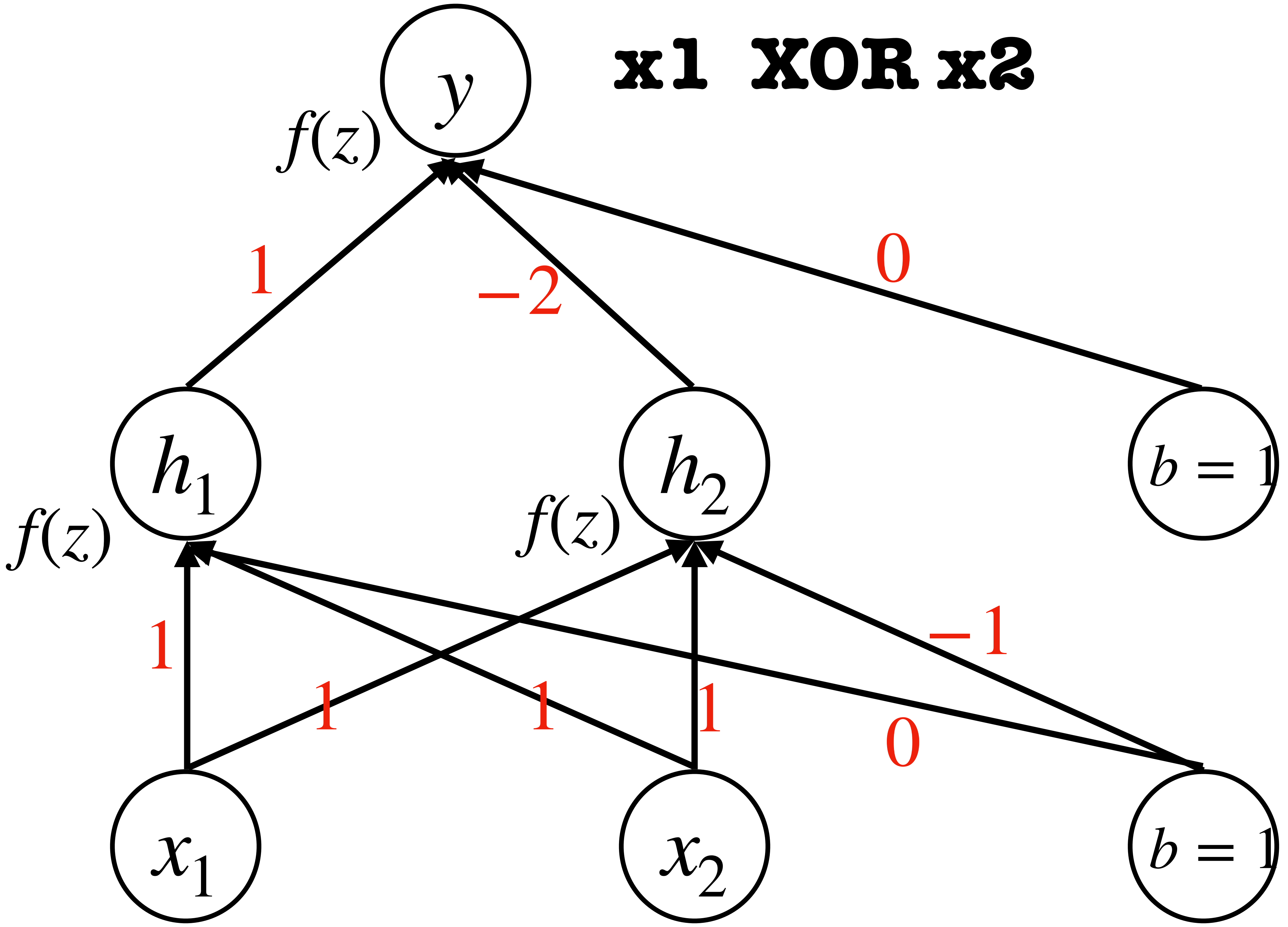

OR

x1	x2	$h1_0$	$h2_0$	h1	h2	y_0	y
0	0	0	-1	0	0	0	0
0	1	1	0	1	0	1	1
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One Solution to the XOR Problem

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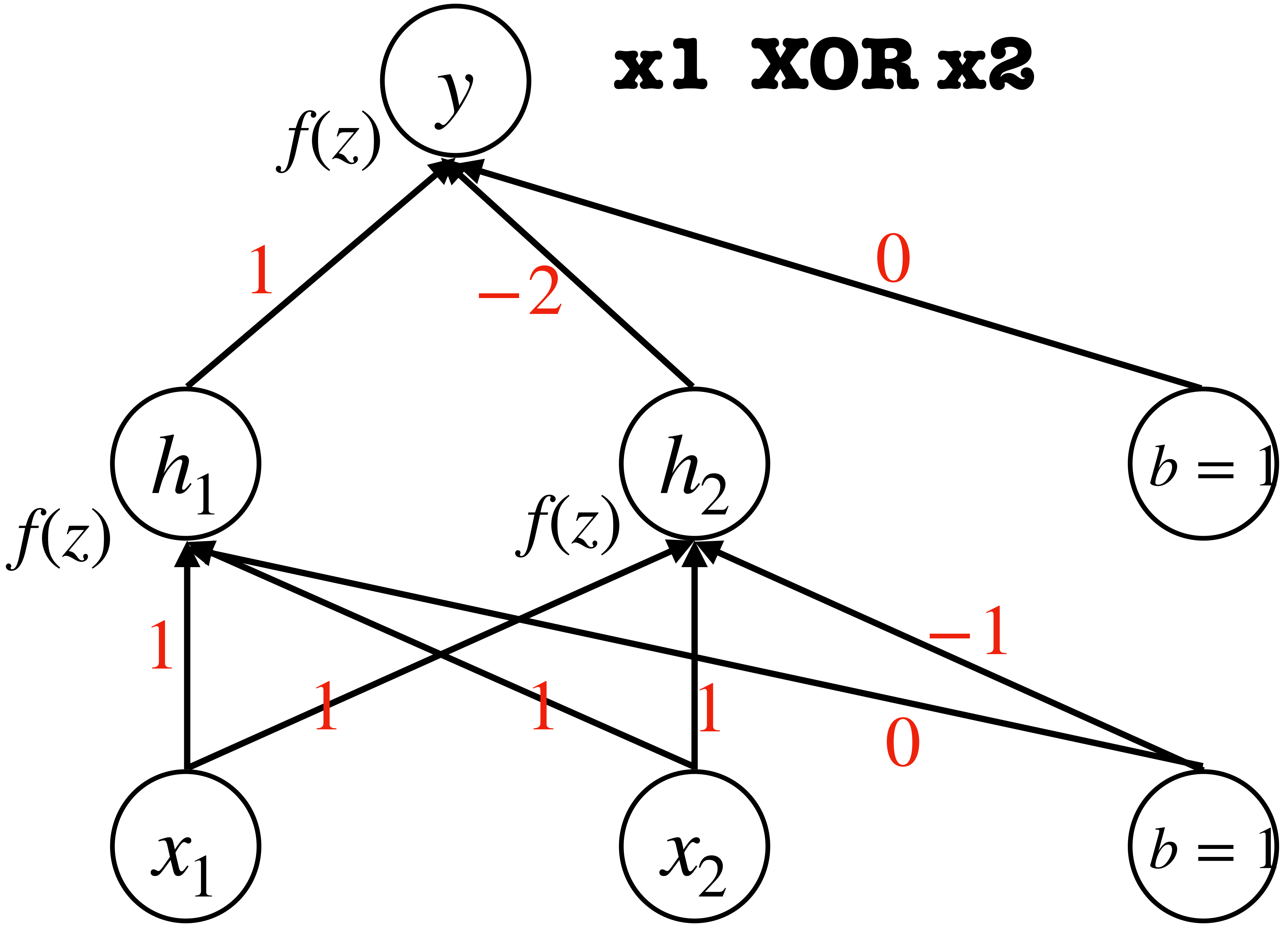

OR AND

x1	x2	$h1_0$	$h2_0$	h1	h2	y_0	y
0	0	0	-1	0	0	0	0
0	1	1	0	1	0	1	1
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One Solution to the XOR Problem

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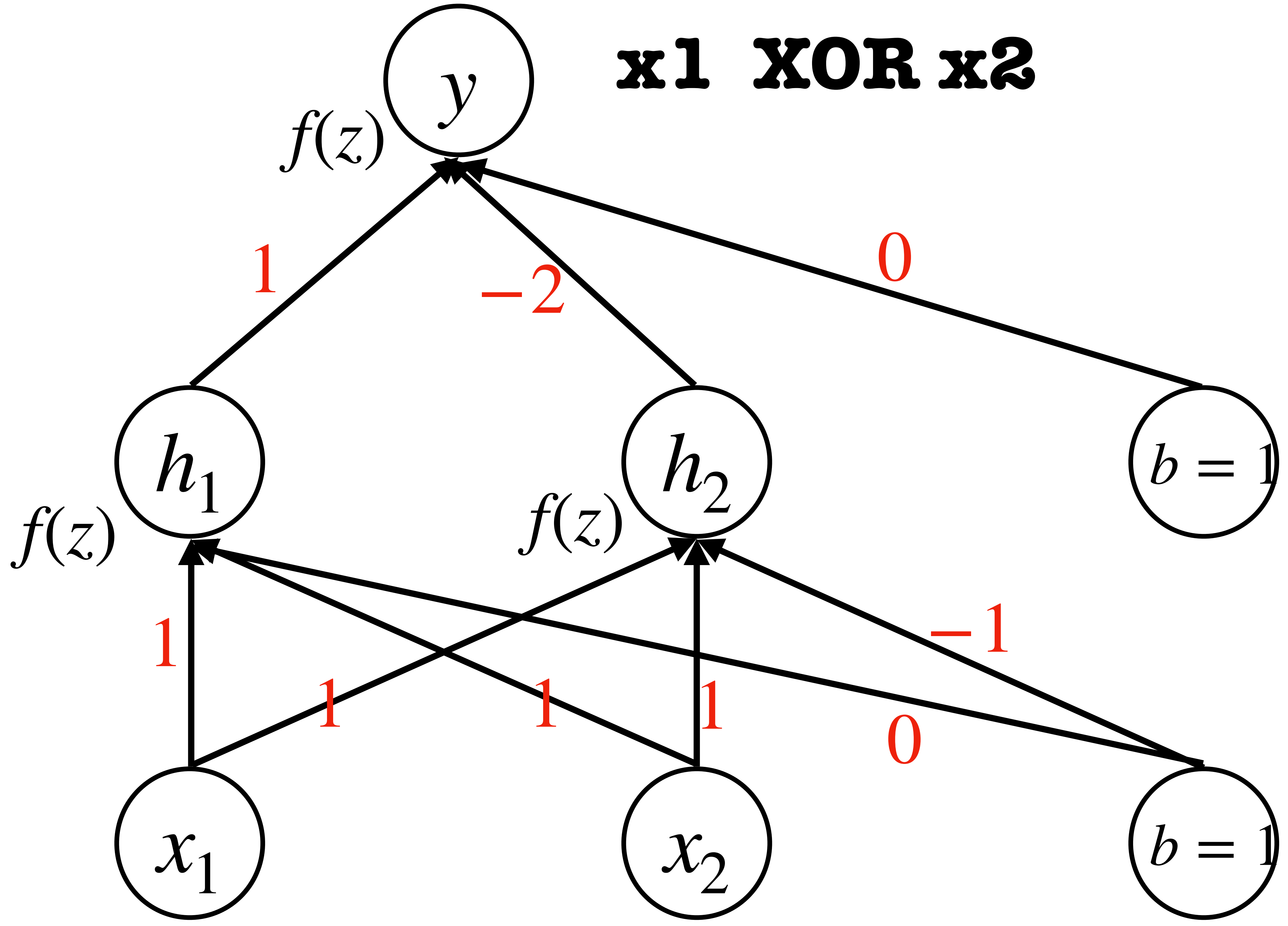
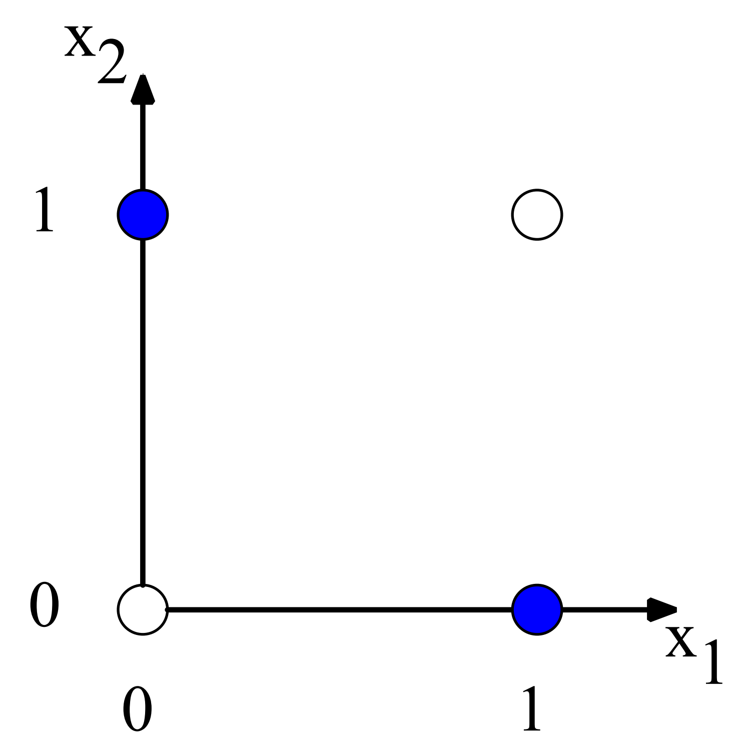


				OR	AND		XOR
x1	**x2**	**$h1_0$**	**$h2_0$**	**h1**	**h2**	**y_0**	**y**
0	0	0	-1	0	0	0	0
0	1	1	0	1	0	1	1
1	0	1	0	1	0	1	1
1	1	2	1	1	1	-1	0

Why Adding a Layer Solves the Problem?

Key Idea = using nonlinear transformation at intermediate stages



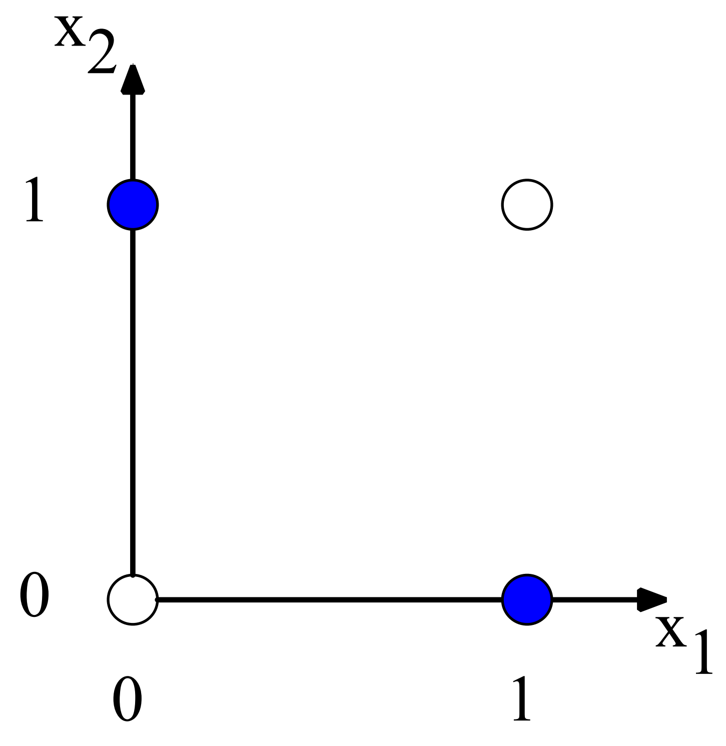
a) The original x space

Why Adding a Layer Solves the Problem?

Key Idea = using nonlinear transformation at intermediate stages

- For a Perceptron, the nonlinearly transformated result directly becomes the output;

➡ *There is no chance for the model to do further computation on it;*



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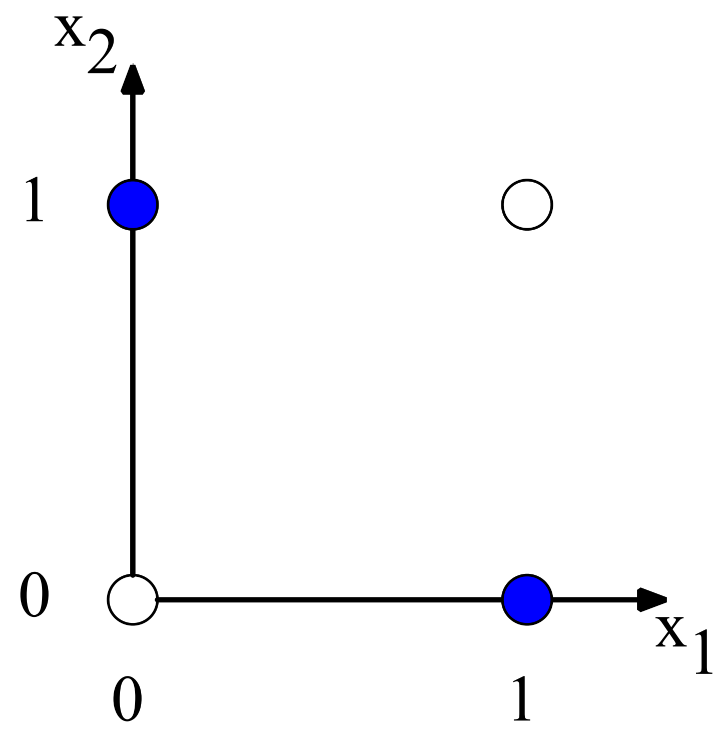

a) The original *x* space

- For a two-layer neural network: it can **use the nonlinear transformation of the first layer to extract whatever useful features** (in our case: **AND** and **OR**);

Why Adding a Layer Solves the Problem?

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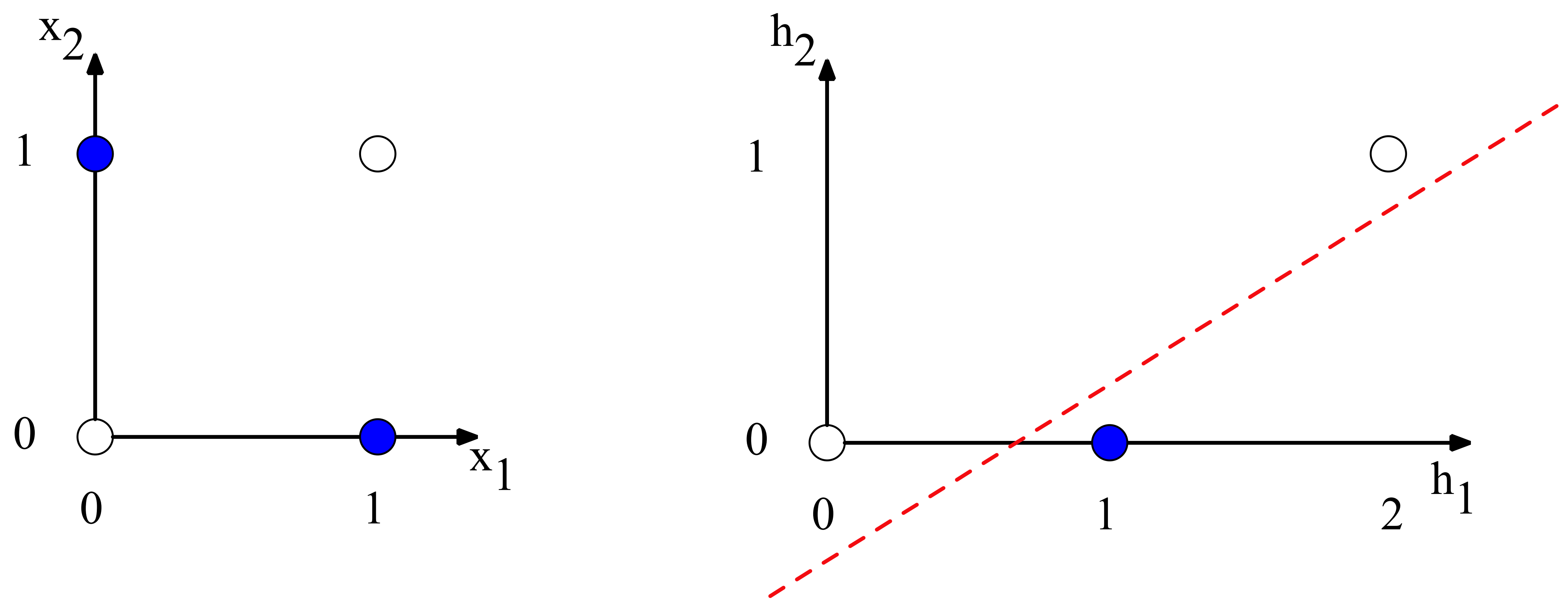


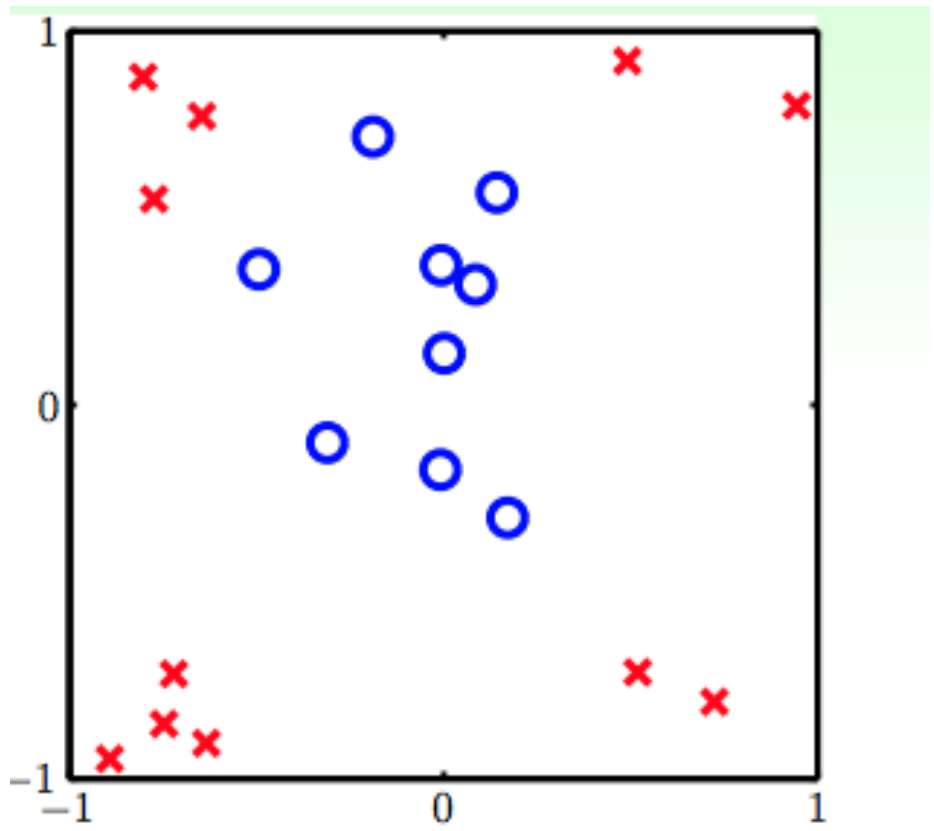
a) The original *x* space   b) The new (linearly separable) *h* space

- For a two-layer neural network: it can **use the nonlinear transformation of the first layer to extract whatever useful features** (in our case: **`AND`** and **`OR`**);

- And it can use the extracted hidden features to do more computation (further nonlinear transformation) — *it could be that those intermediate features are transformed to a linearly separable space!*

More on Nonlinearity

Nonlinear transformation are extremely powerful



https://web.cs.ucla.edu/~chohsieh/teaching/CS260_Winter2019/lecture6.pdf

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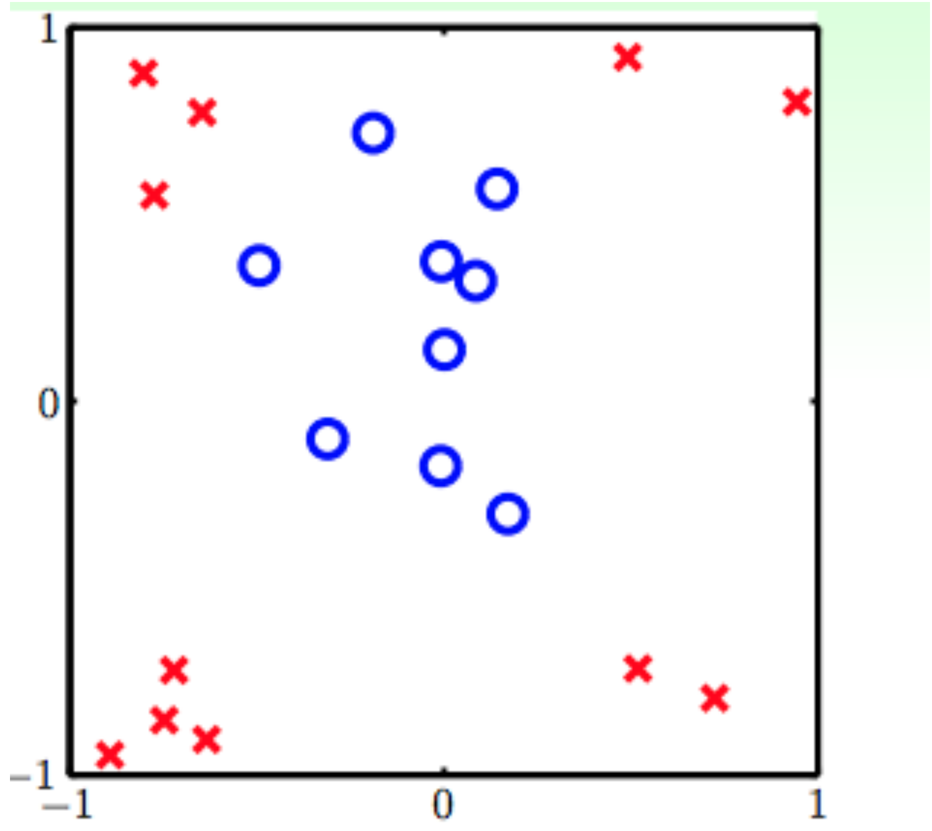

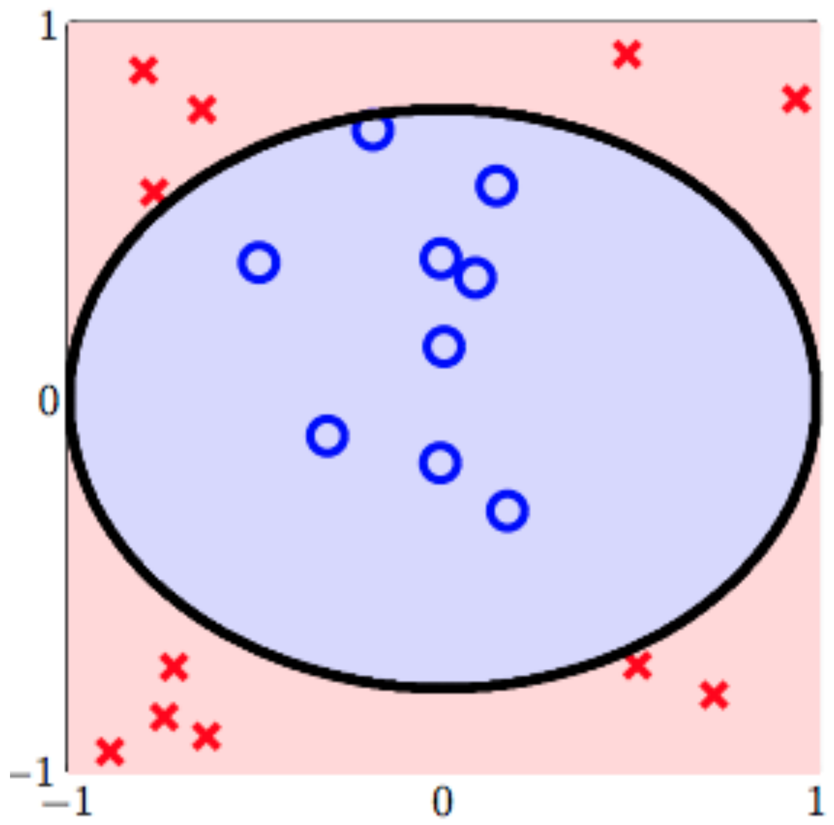

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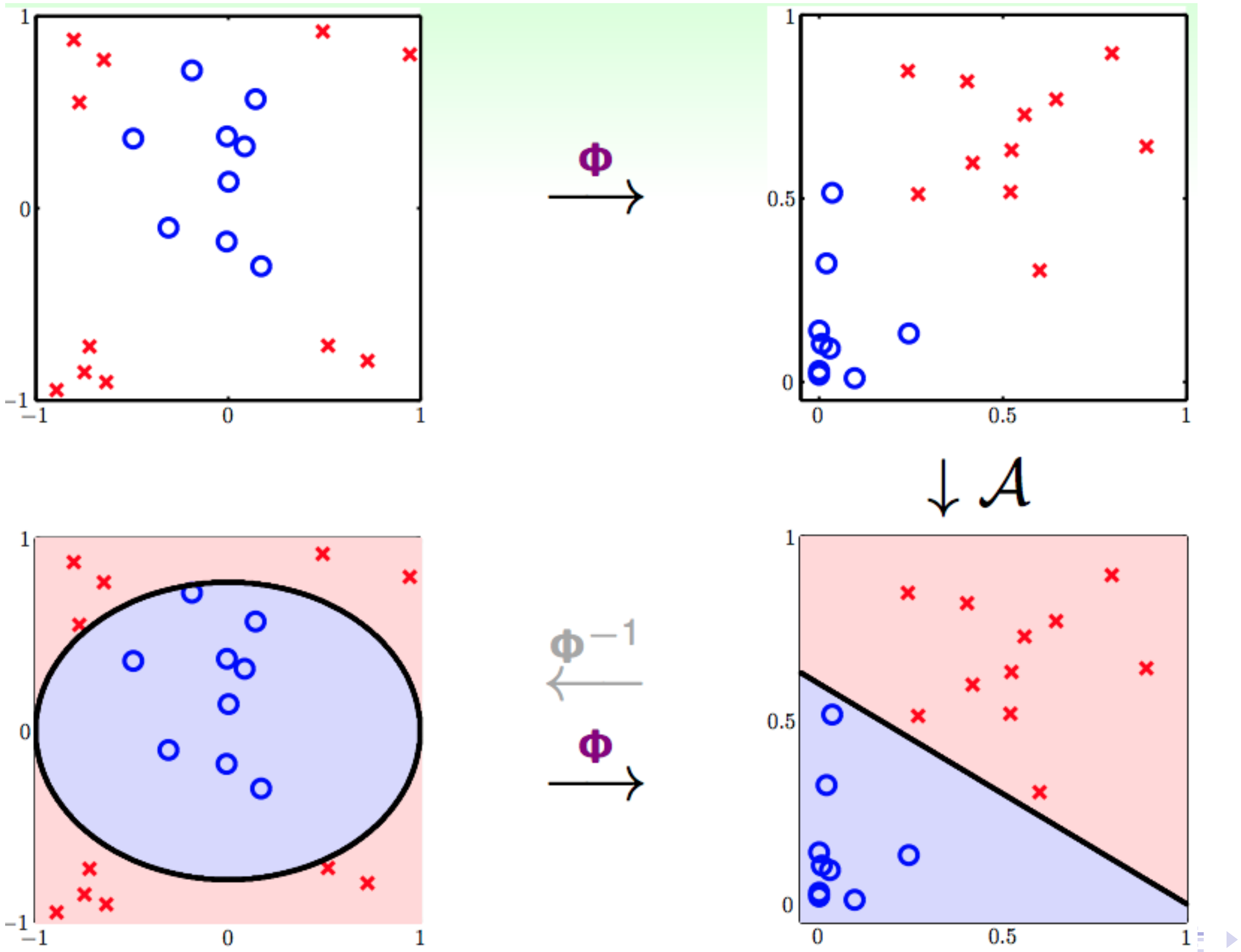


e.g. Tranforming to the Polar coordinate to make an originally non-linearly-separable dataset separable!

https://web.cs.ucla.edu/~chohsieh/teaching/CS260_Winter2019/lecture6.pdf

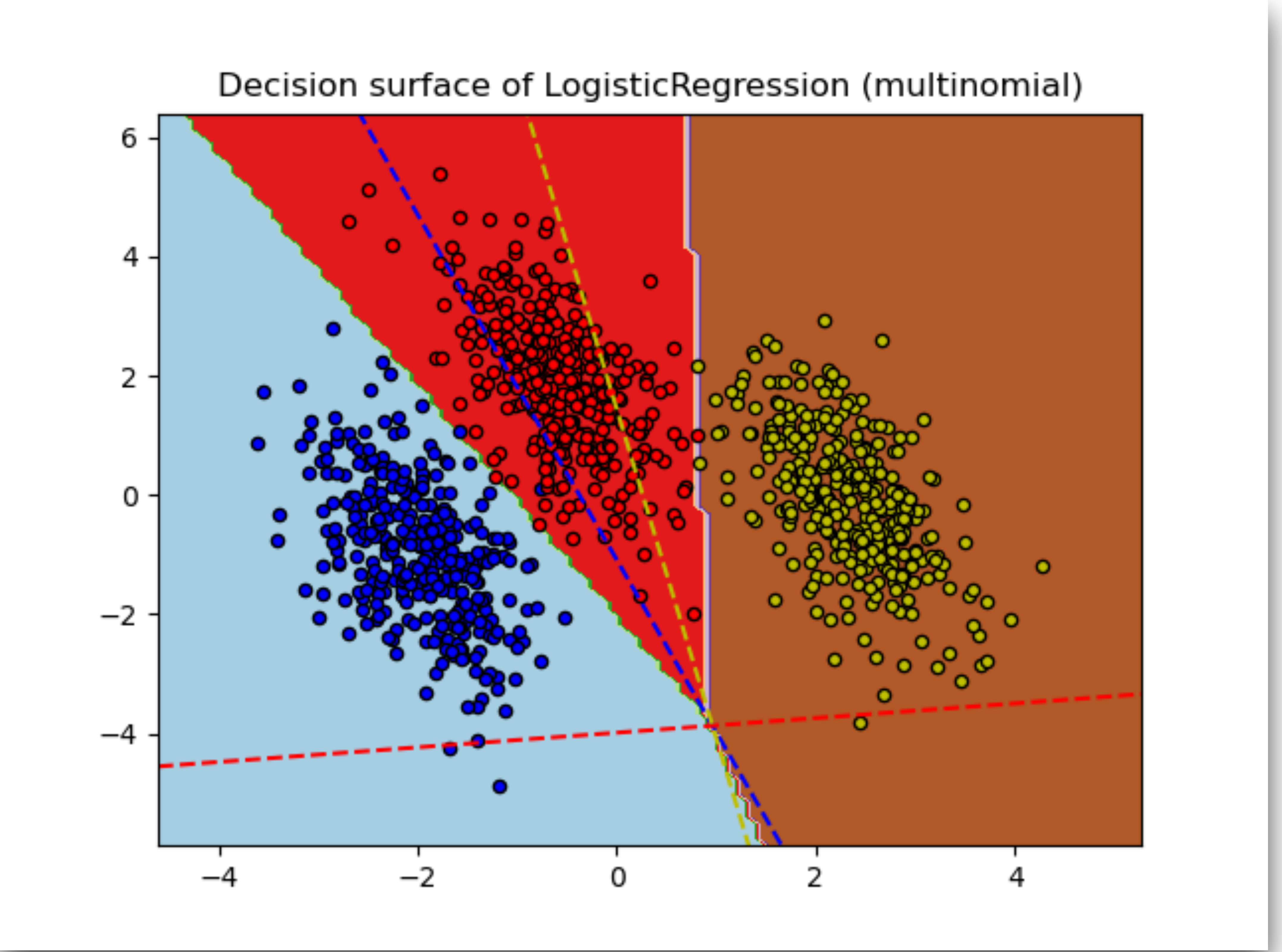
Generalizing the Architecture

From Logistic Regression to Multinomial Logistic Regression

Generalizing the Architecture

From Logistic Regression to Multinomial Logistic Regression

- Let's consider a simple generalization: imagine we are doing a **classification** task with **multiple classes / labels**. How to represent multiple outcomes?



Generalizing the Architecture

From Logistic Regression to Multinomial Logistic Regression

- Let's consider a simple generalization: imagine we are doing a **classification** task with **multiple classes / labels**. How to represent multiple outcomes?

- Two tweaks:

  - Have multiuple output nodes (one per class);

  - Switch the activation function to ***softmax***;

Generalizing the Architecture

From Logistic Regression to Multinomial Logistic Regression

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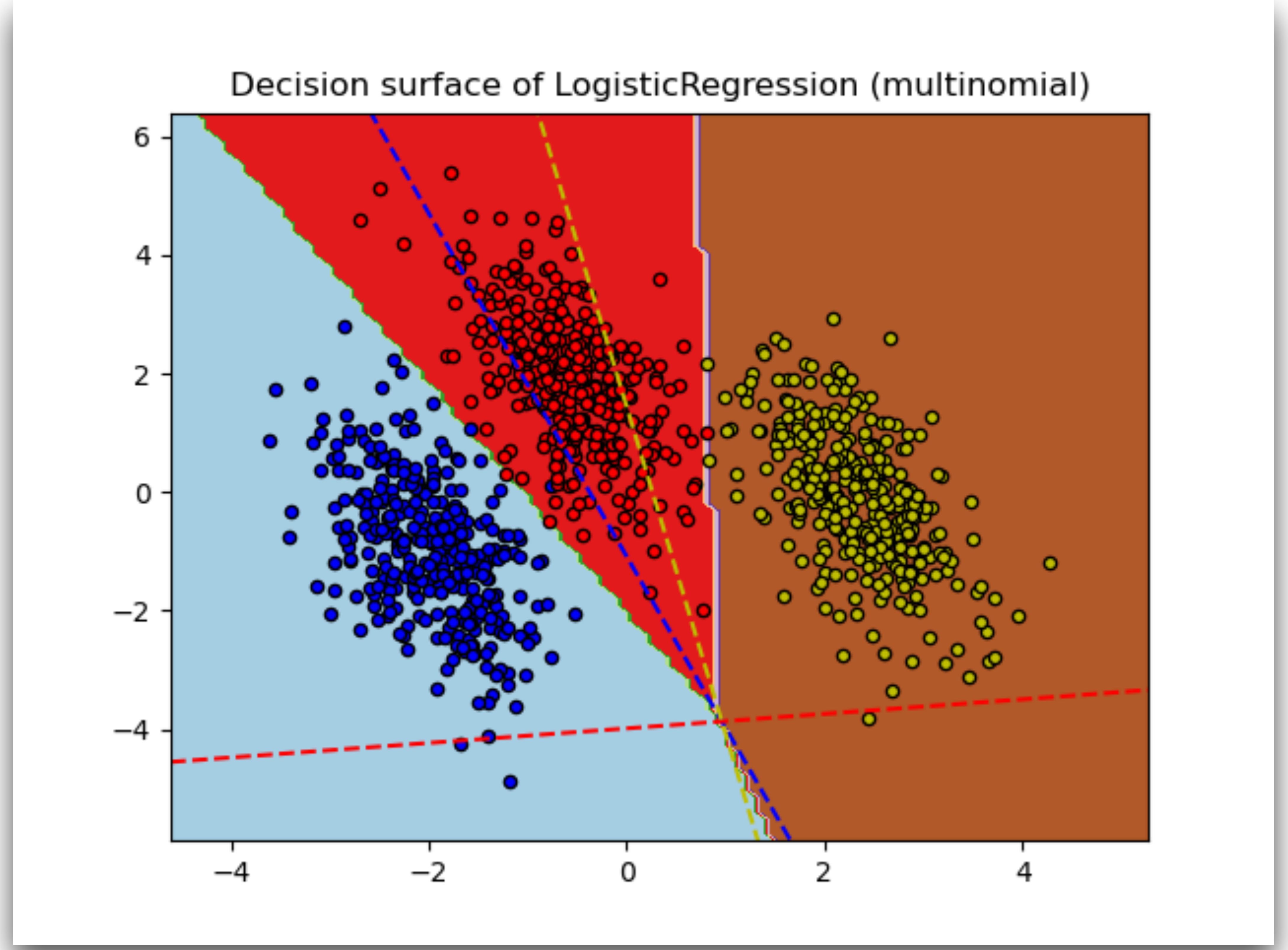
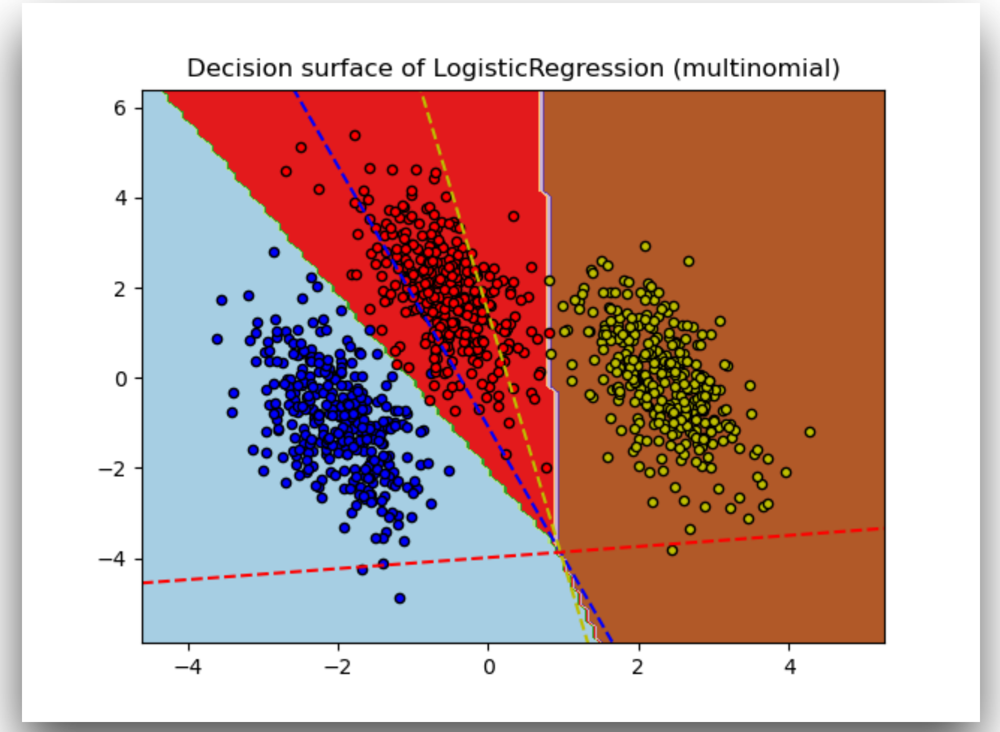
- This is also known as **multinomial logistic regression**.



Generalizing the Architecture

Multinomial Logistic Regression

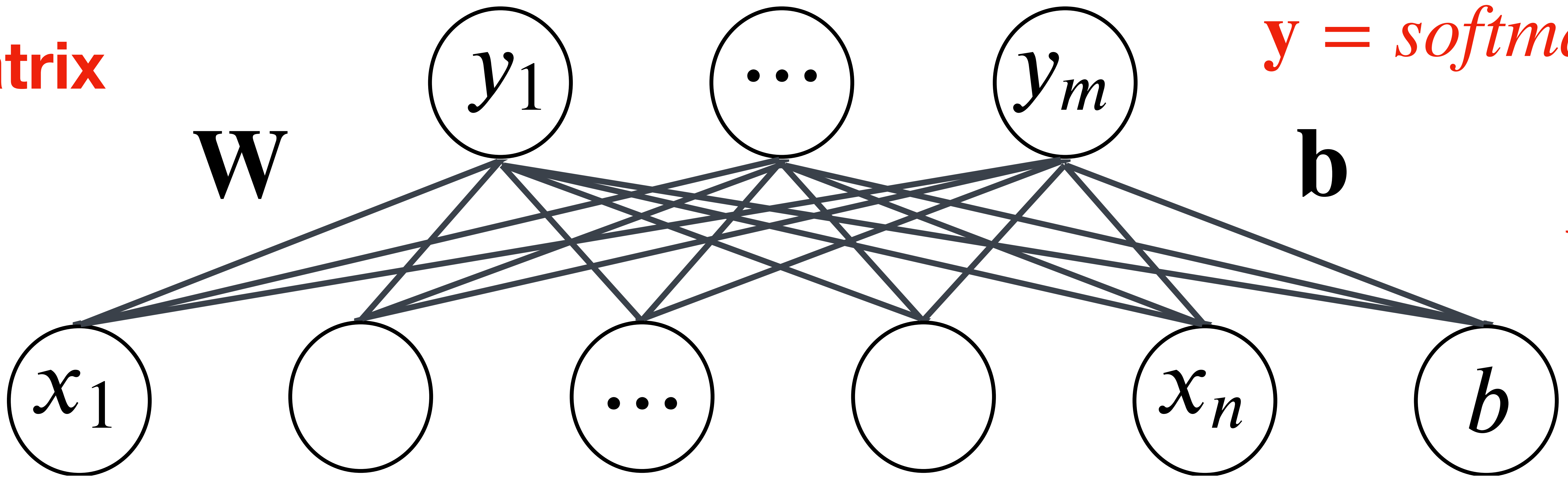


- Suppose we have n input features and m classes;
- One way to represent it is a 1-layer neural network with multiple output nodes (i.e., a layer of neurons)
- We don't count the input layer as a layer, so this network has 1 layer.

$\mathbf{y} = [y_1, \cdots, y_m]$ **is a vector:**

$\mathbf{y} = softmax(\mathbf{Wx} + \mathbf{b})$

W is a matrix



b is a vector

Input a list of scalars

Softmax: A generalization of Sigmoid

Getting a probability distribution for multi-class classification!

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- Formally: for a vector z of dimensionality k:

$$softmax(\mathbf{z}) = \left[\frac{\exp(z_1)}{\sum_{i=1}^{k} \exp(z_i)}, \frac{\exp(z_2)}{\sum_{i=1}^{k} \exp(z_i)}, \ldots, \frac{\exp(z_k)}{\sum_{i=1}^{k} \exp(z_i)} \right]$$

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- An example:

$$\mathbf{z} = [0.6, 1.1, -1.5, 1.2, 3.2, -1.1]$$

$$softmax(\mathbf{z}) = [0.055, 0.090, 0.006, 0.099, 0.74, 0.010]$$

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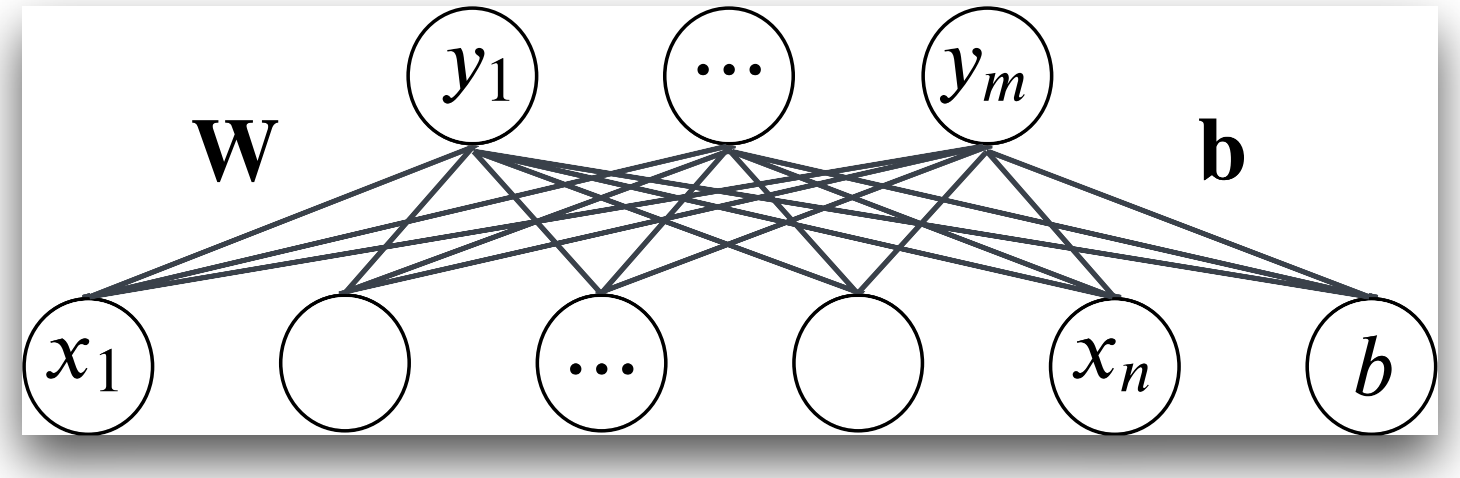
Generalizing the Notations

Representing a list of neurons



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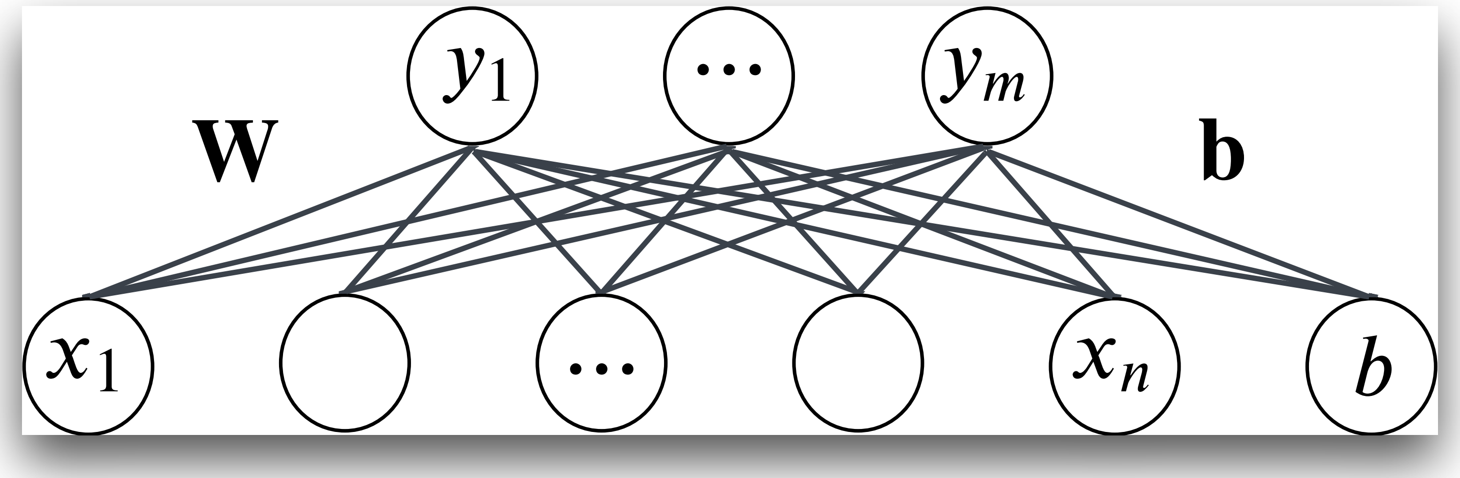

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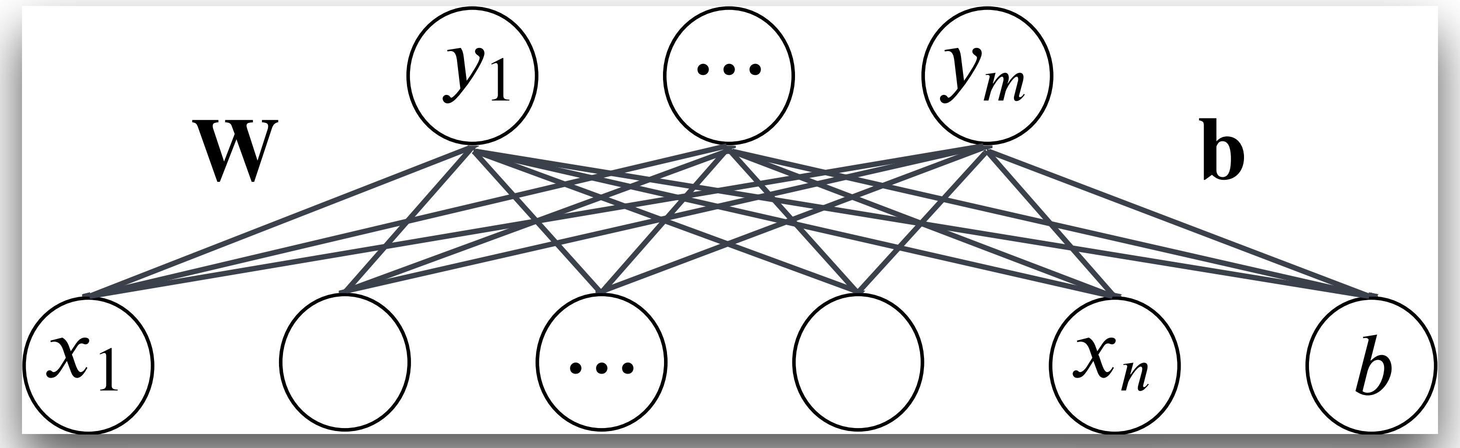
Generalizing the Notations



Representing a list of neurons

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- Now, we have multiple neurons in the layer, and each neuron is connected to all the inputs, so we (naturally) need a matrix to represent the weights.

Generalizing the Notations

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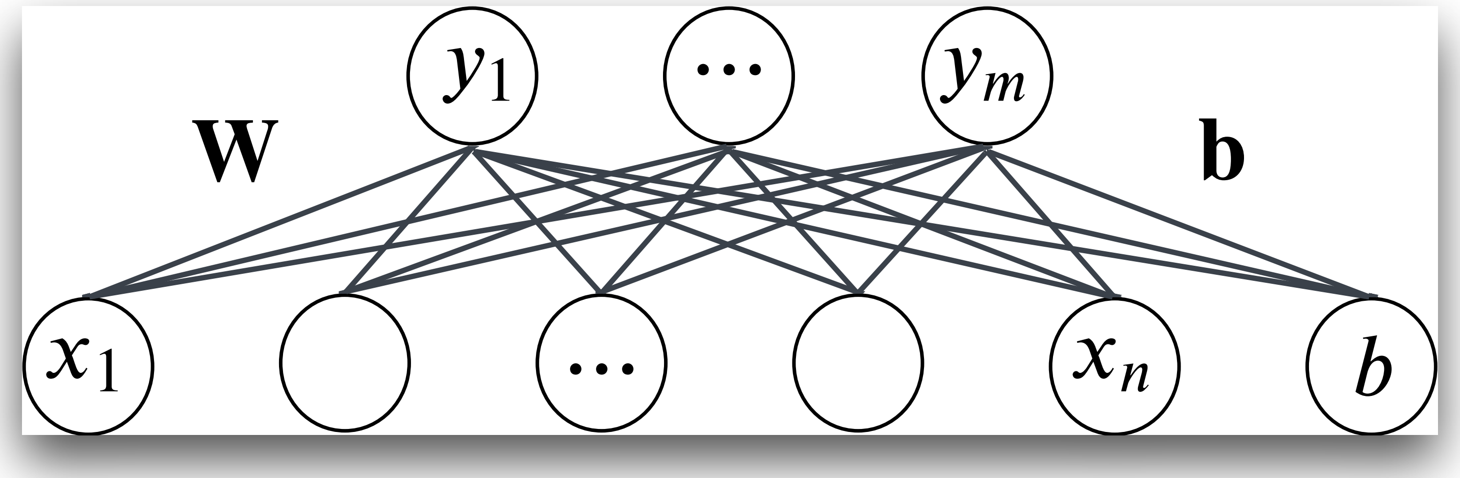

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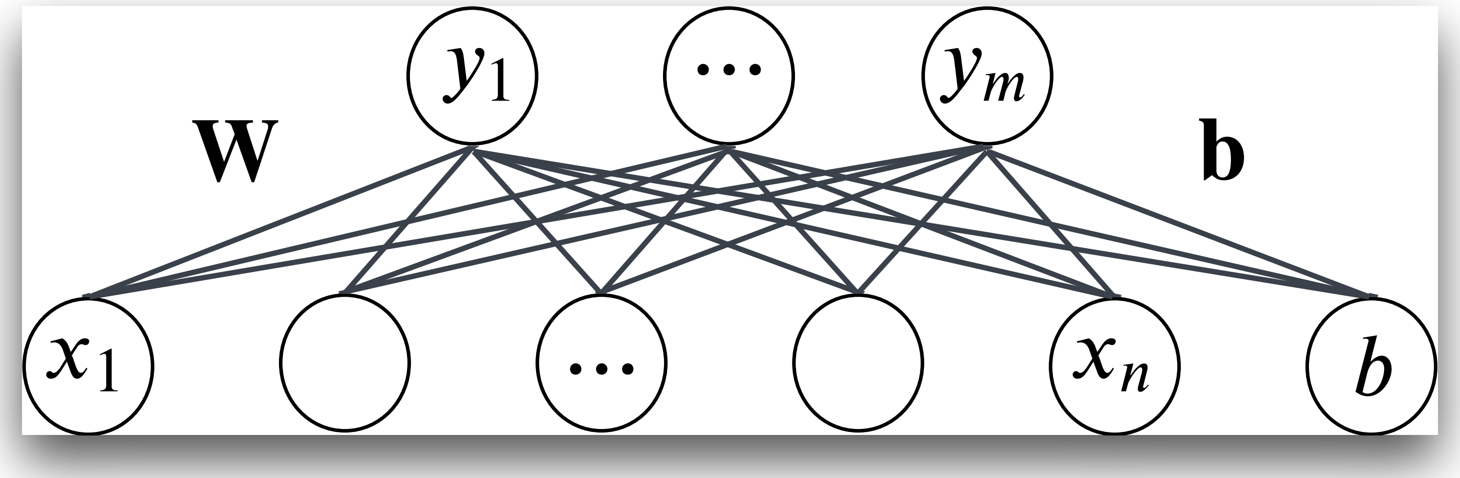

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Generalizing the Notations

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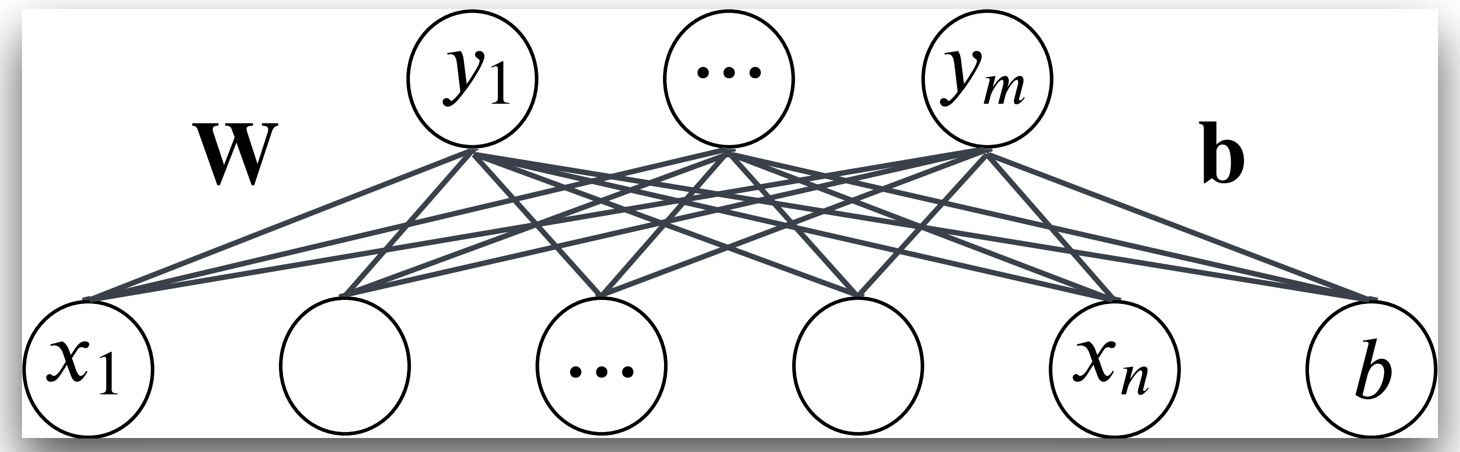

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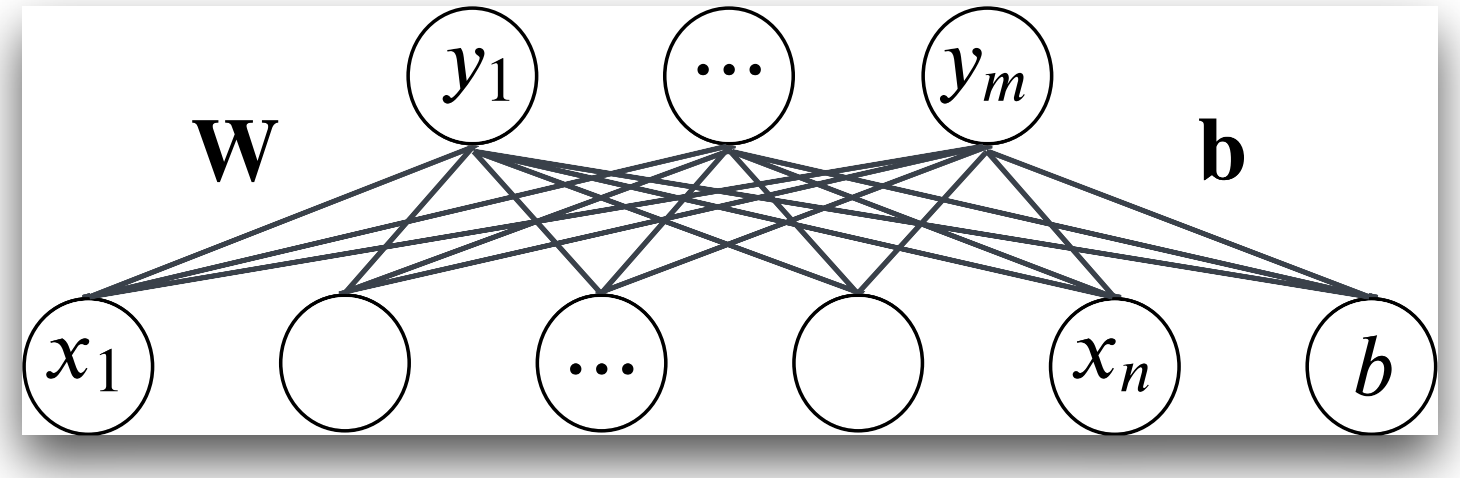

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Generalizing the Notations

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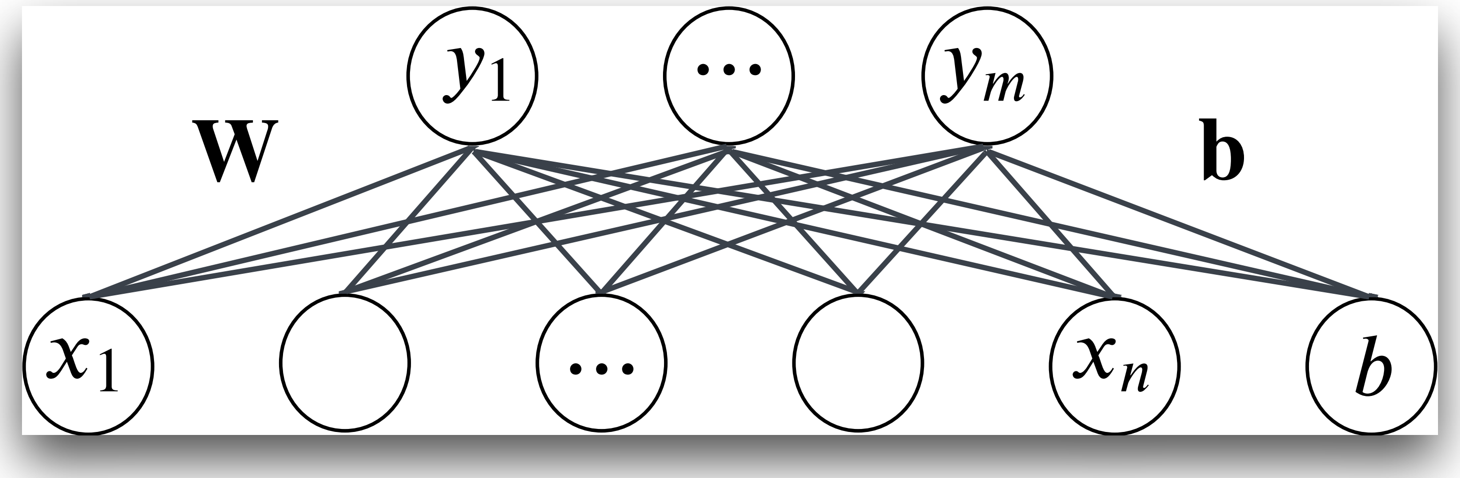

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Generalizing the Notations

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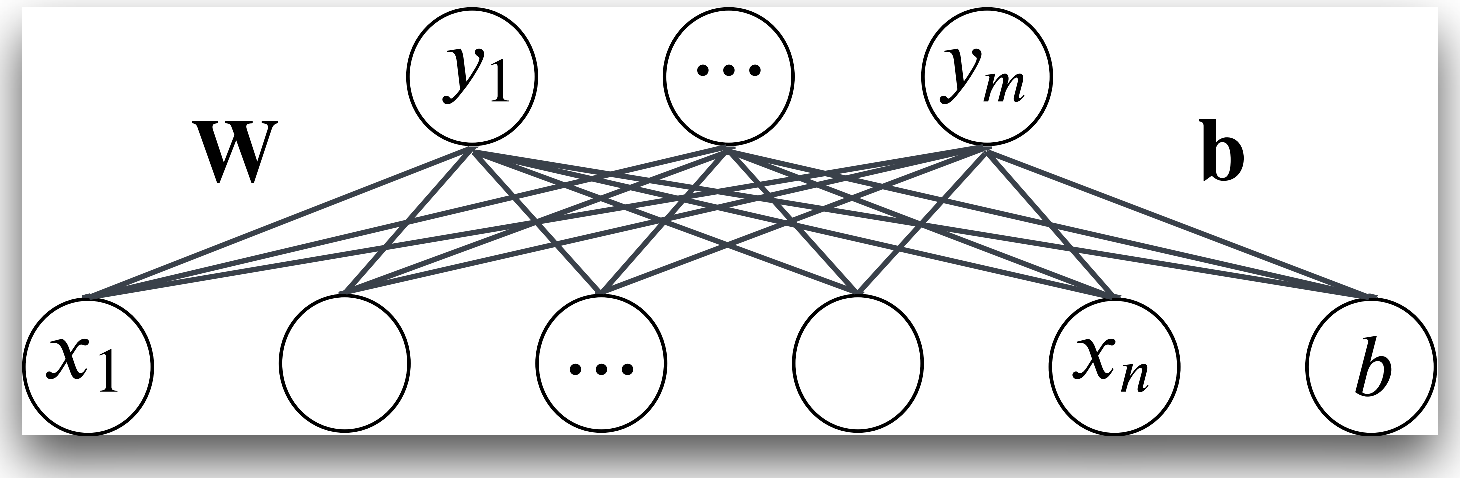

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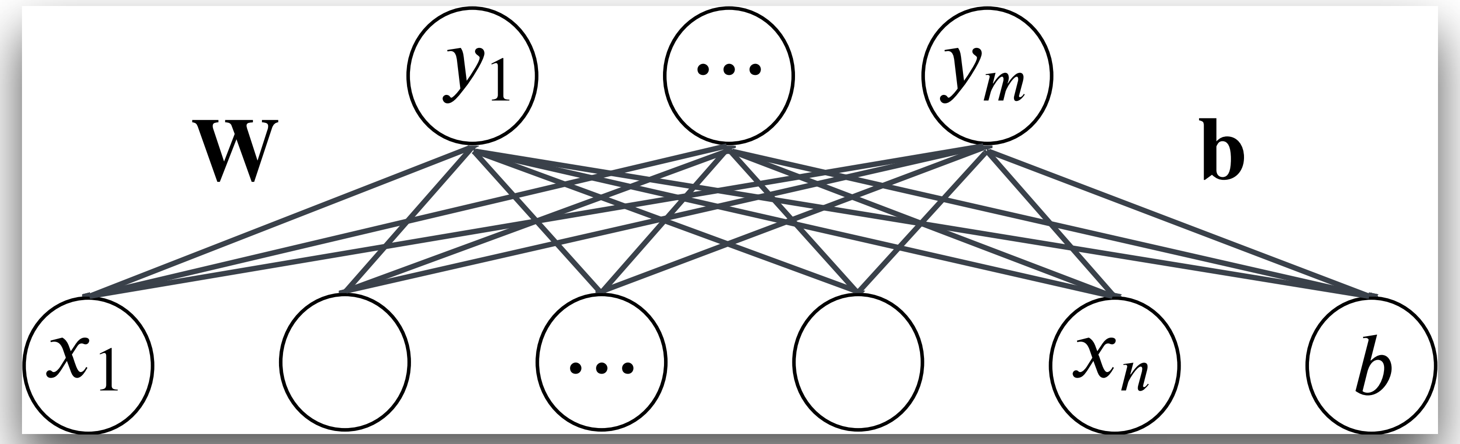
Generalizing the Notations

Representing a list of neurons



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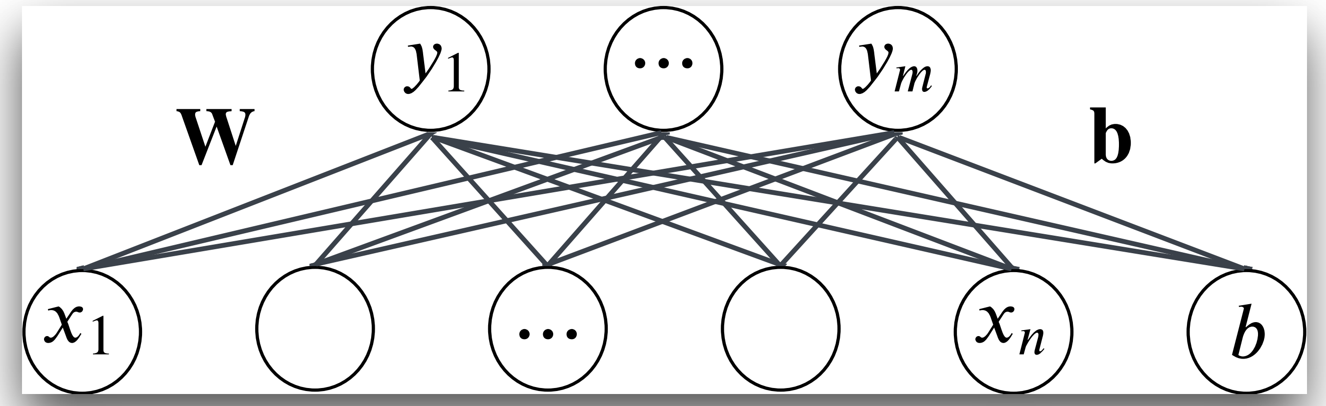
In general: position $[i, j]$ in $\mathbf{W}$ is the weight connecting input j to neuron i.

Generalizing the Notations, cont.



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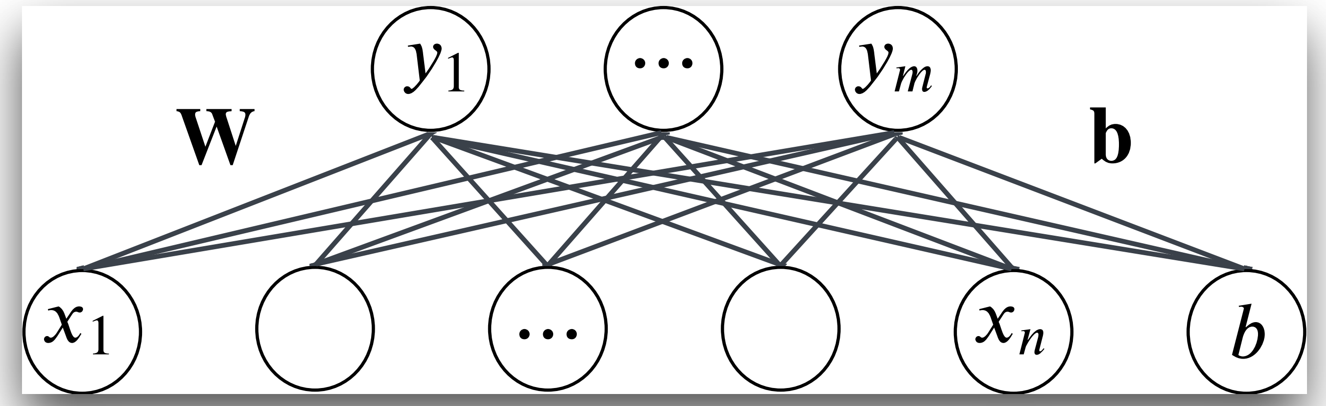

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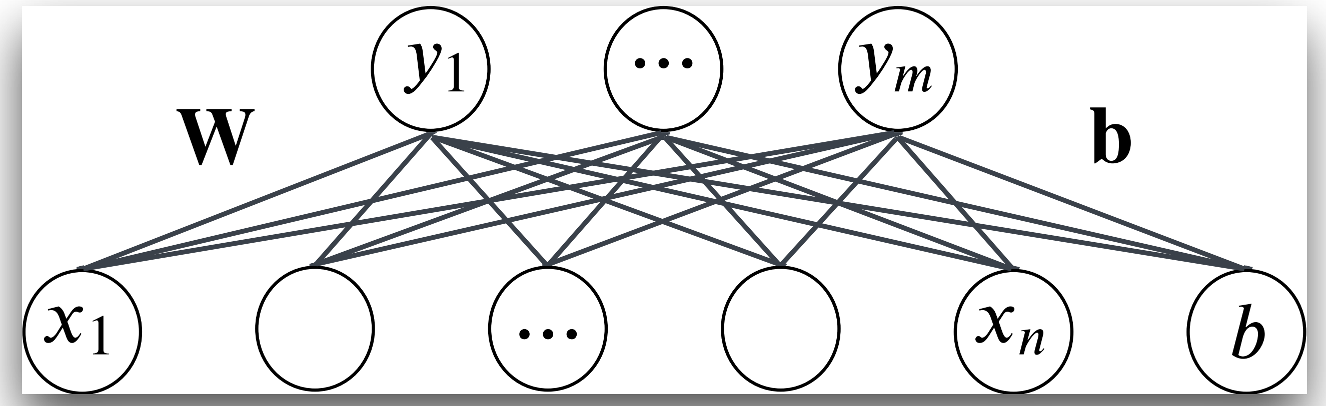

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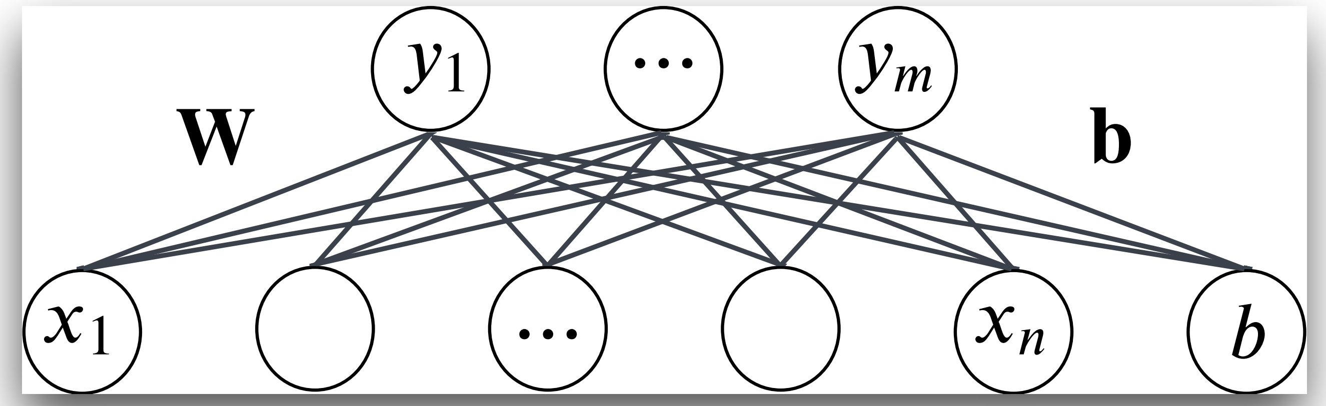

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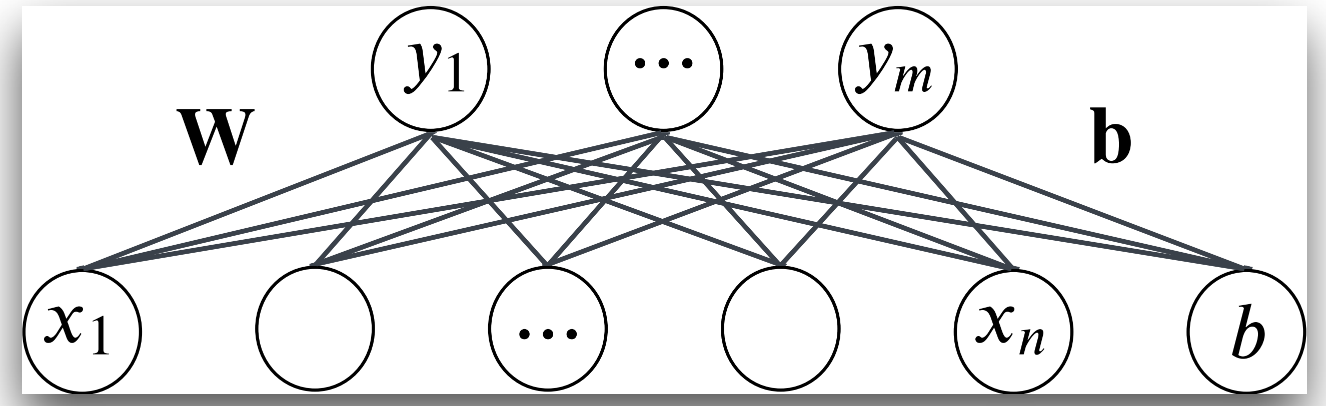

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Generalizing the Notations, cont.

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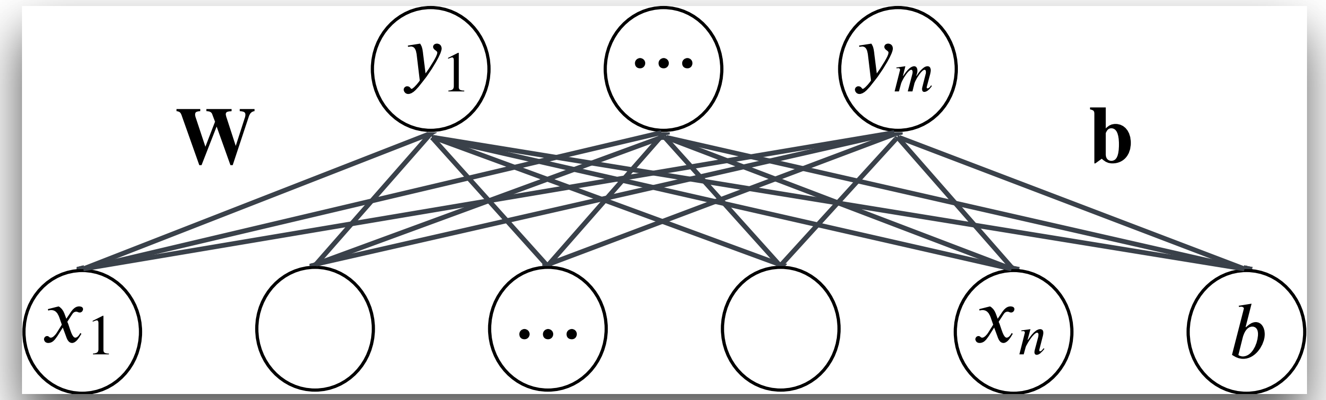

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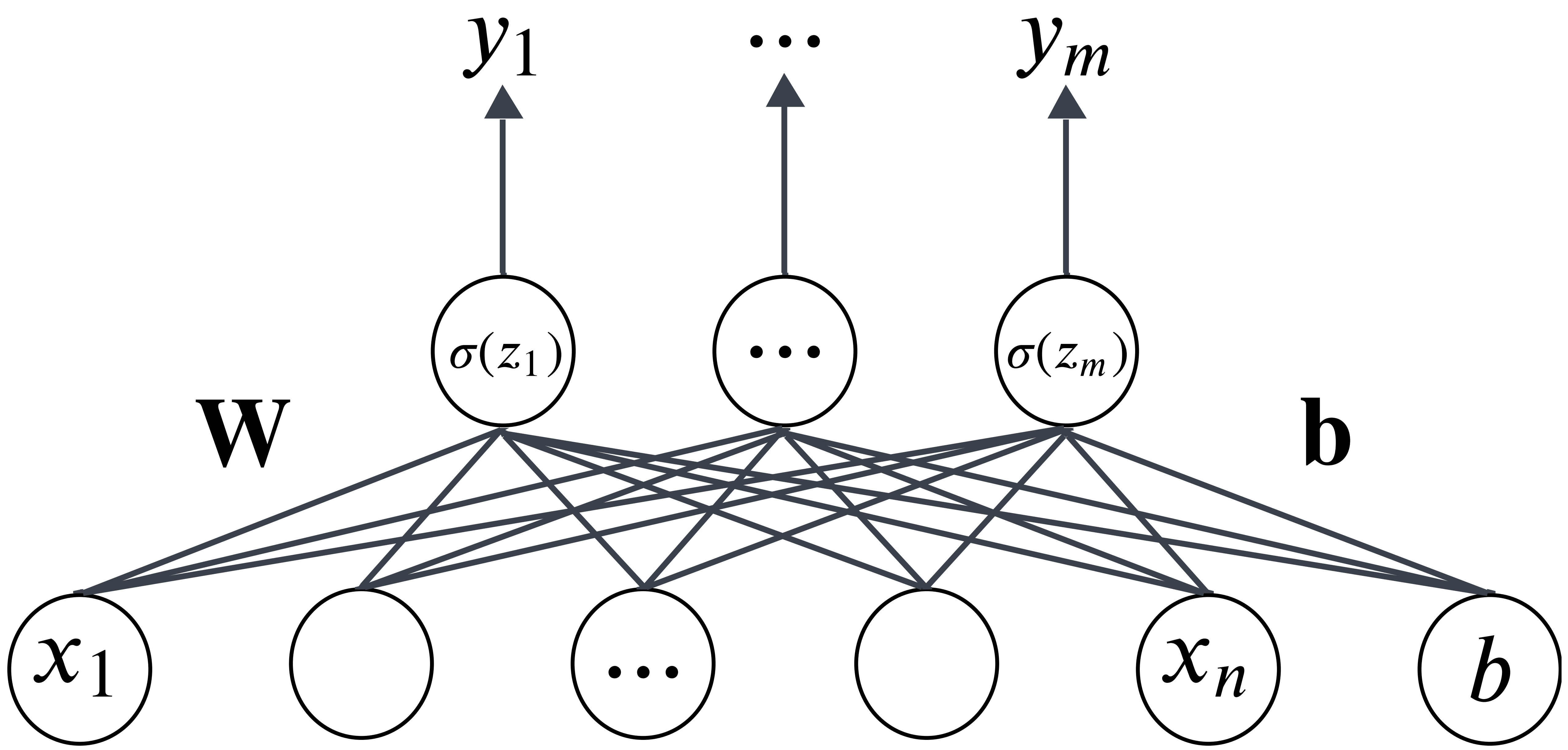
Add $\mathbf{b}$ as another column of $\mathbf{W}$, and stick a 1 to the input vector $\mathbf{x}$.

Generalizing the Notations, cont.

Folding the nonlinear transformation into the neurons



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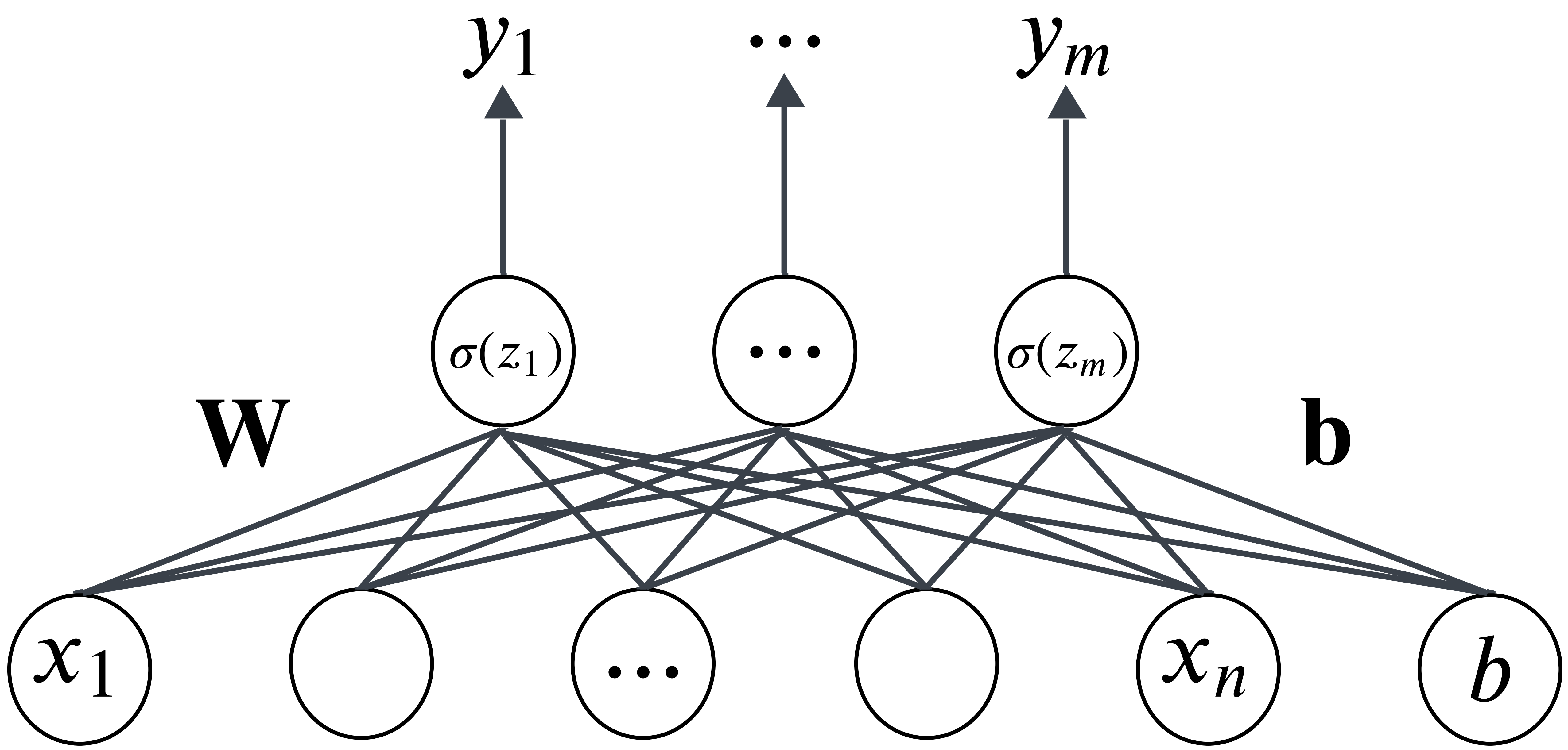
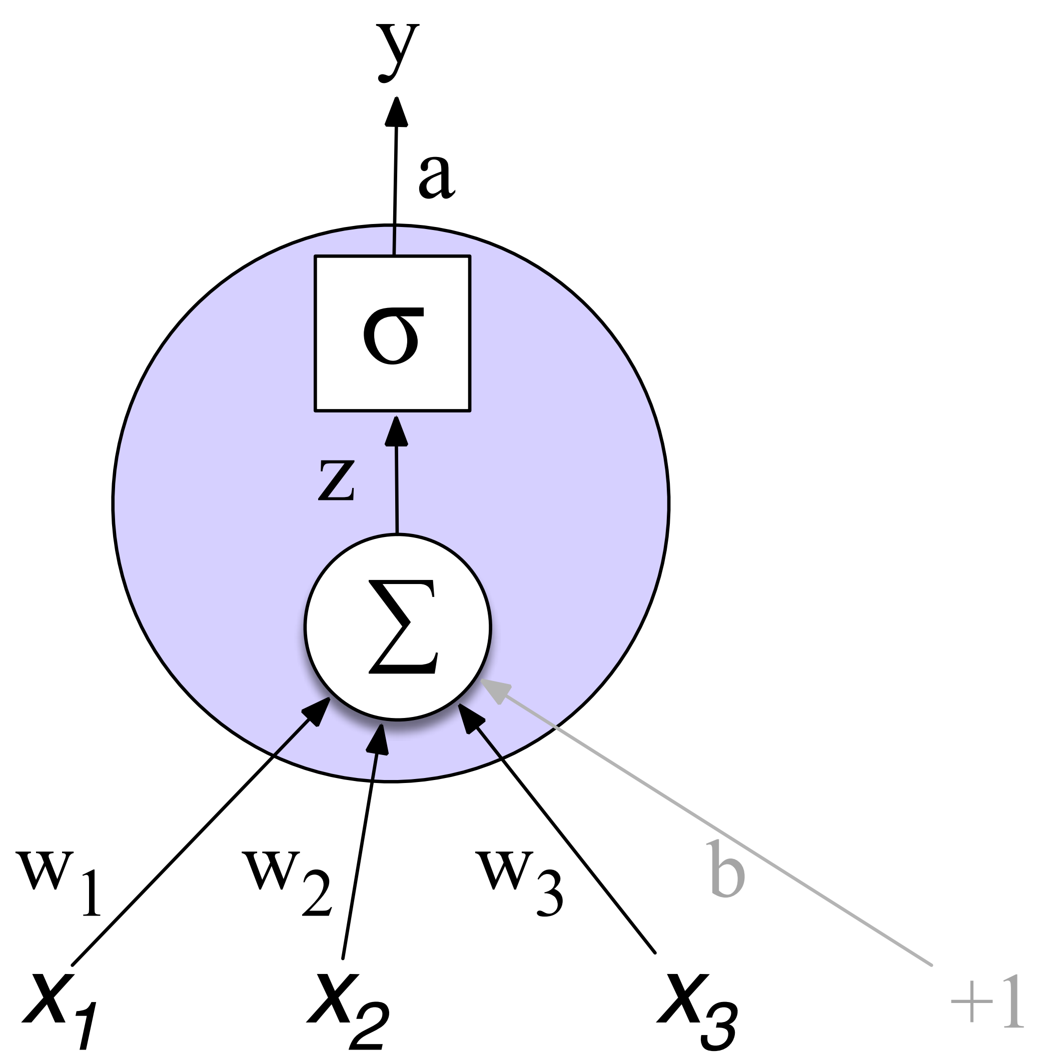


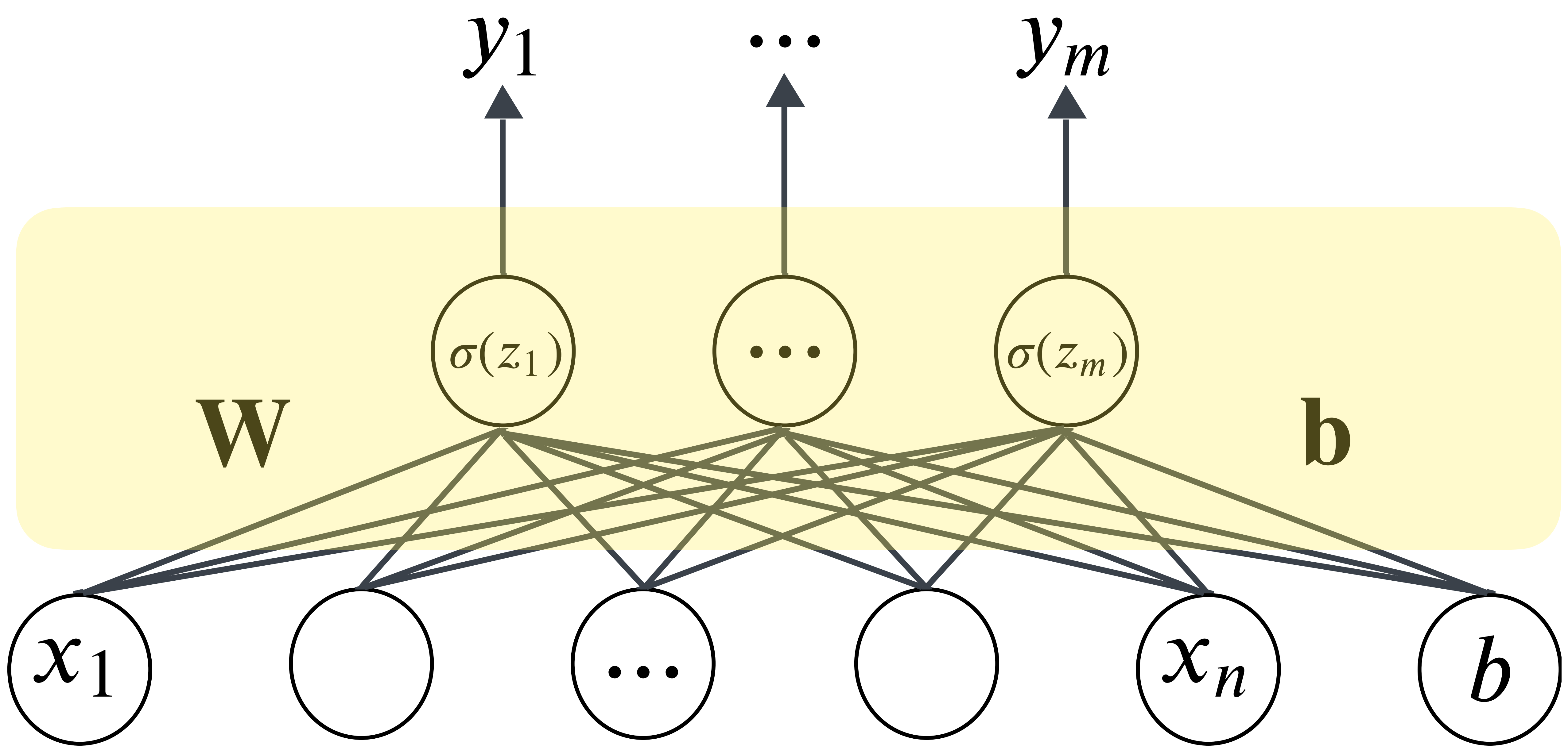


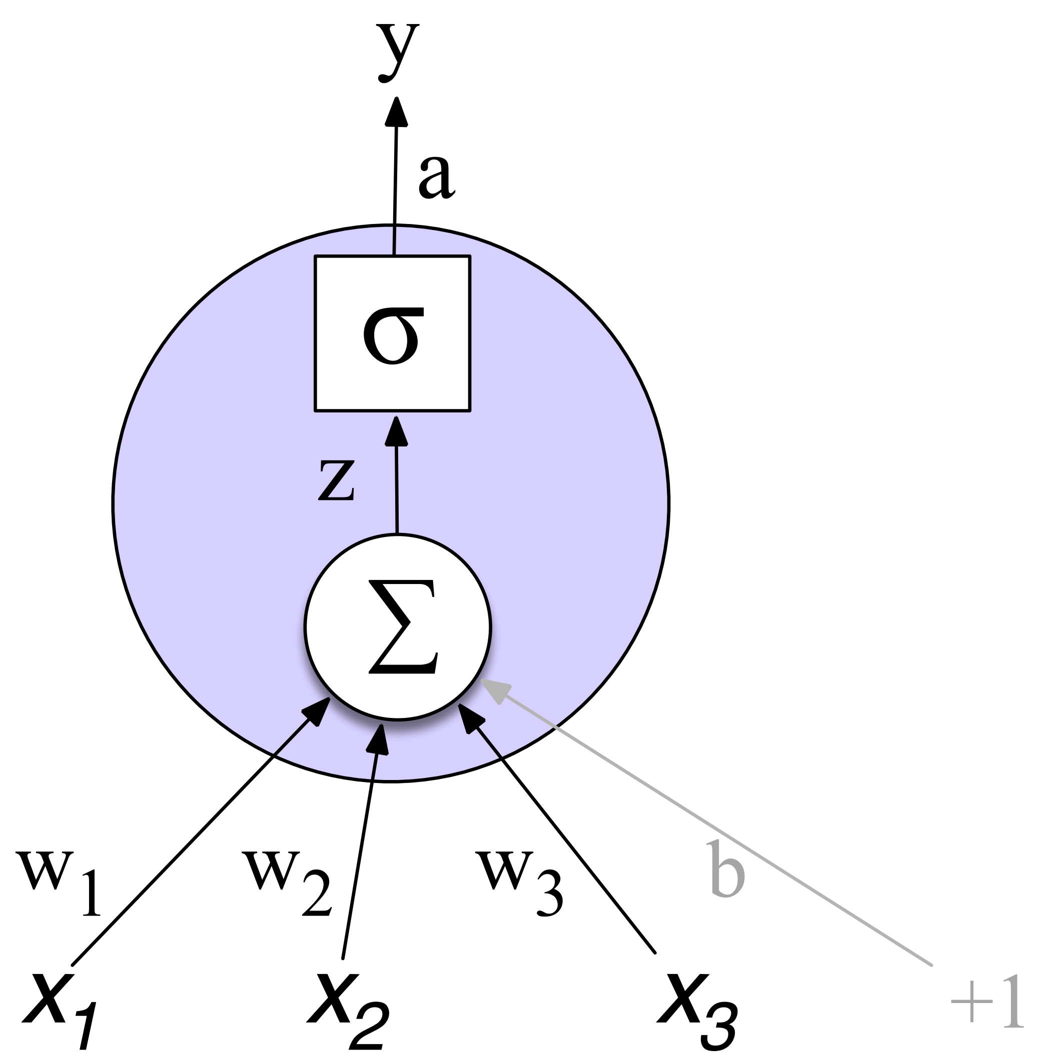
- A neuron in a neural network computes the linearly transformed weighted sum: $f(\mathbf{W}\mathbf{x} + \mathbf{b})$, where f can be softmax, sigmoid, ReLU, tanh, etc.

Generalizing the Notations, cont.

Folding the nonlinear transformation into the neurons

Layer 1





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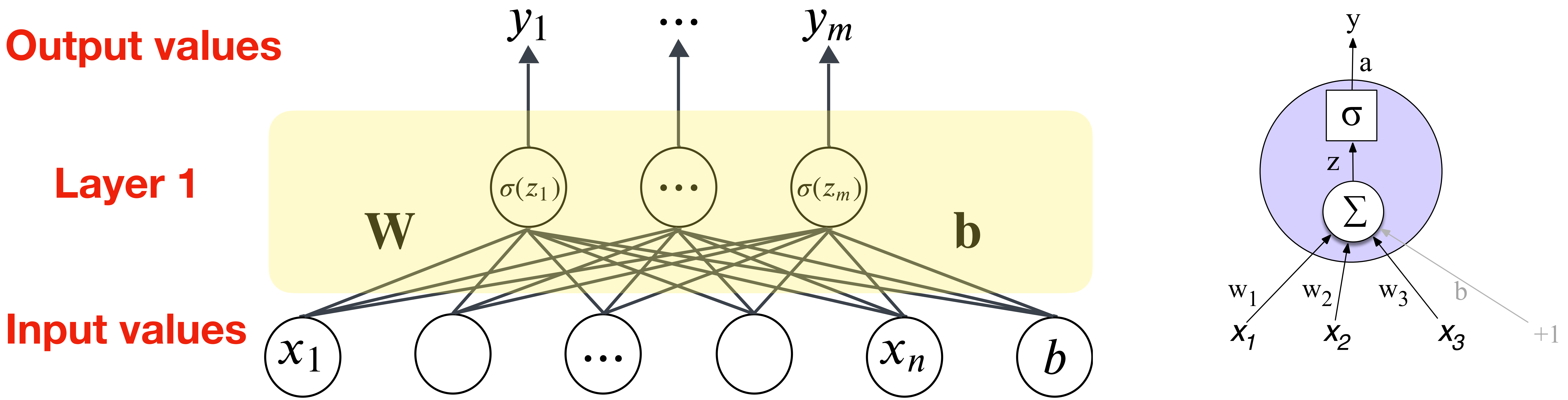
Generalizing the Notations, cont.

Folding the nonlinear transformation into the neurons



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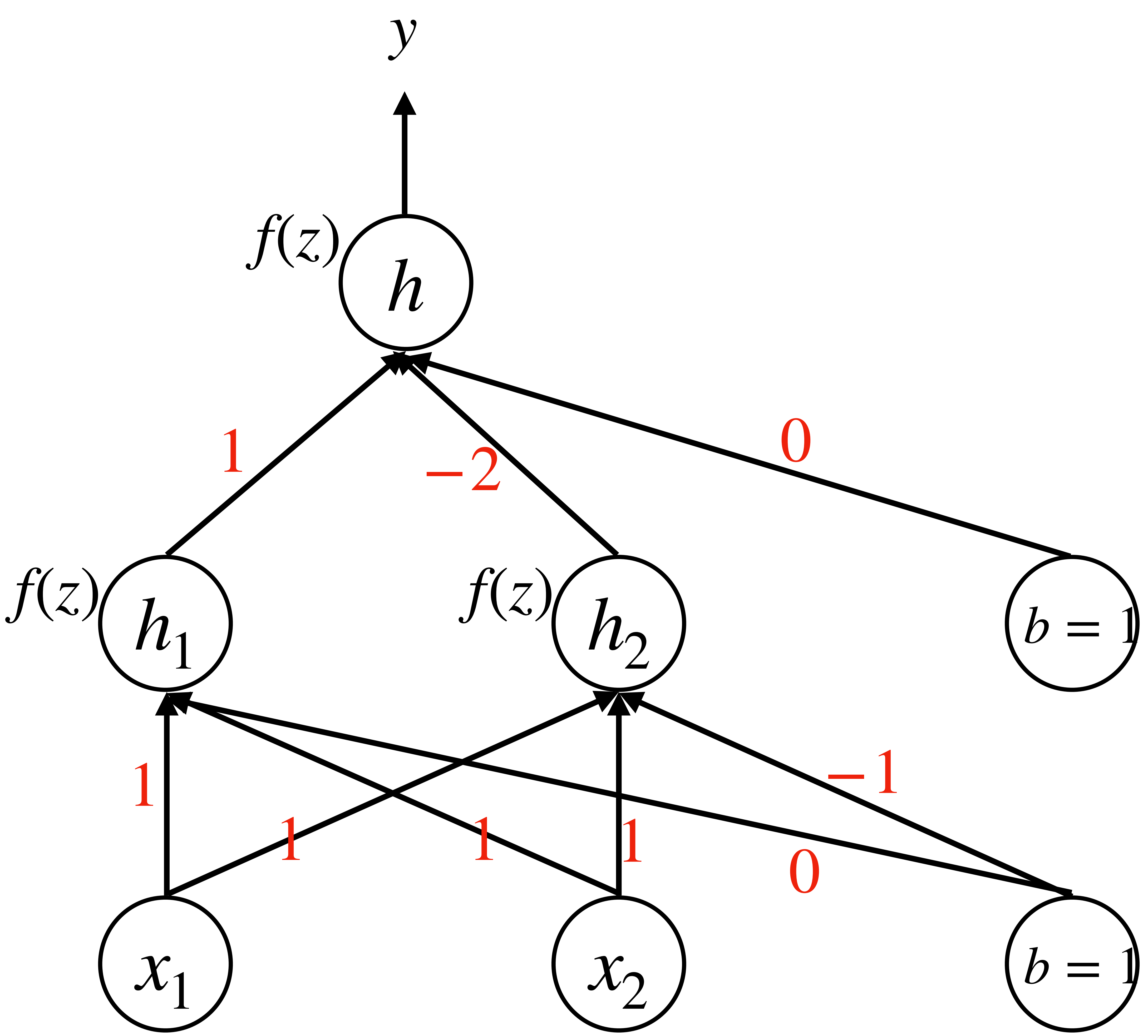
- A layer is a list of neurons, defined by $\mathbf{W}$, $\mathbf{b}$, and the nonlinear transformation.
- Input nodes ($\mathbf{x}$) don’t count as a layer; and output values also don’t.

Generalizing the Notations, cont.

Our XOR network is a two-layer neural network:



Output values

Input values

Generalizing the Notations, cont.

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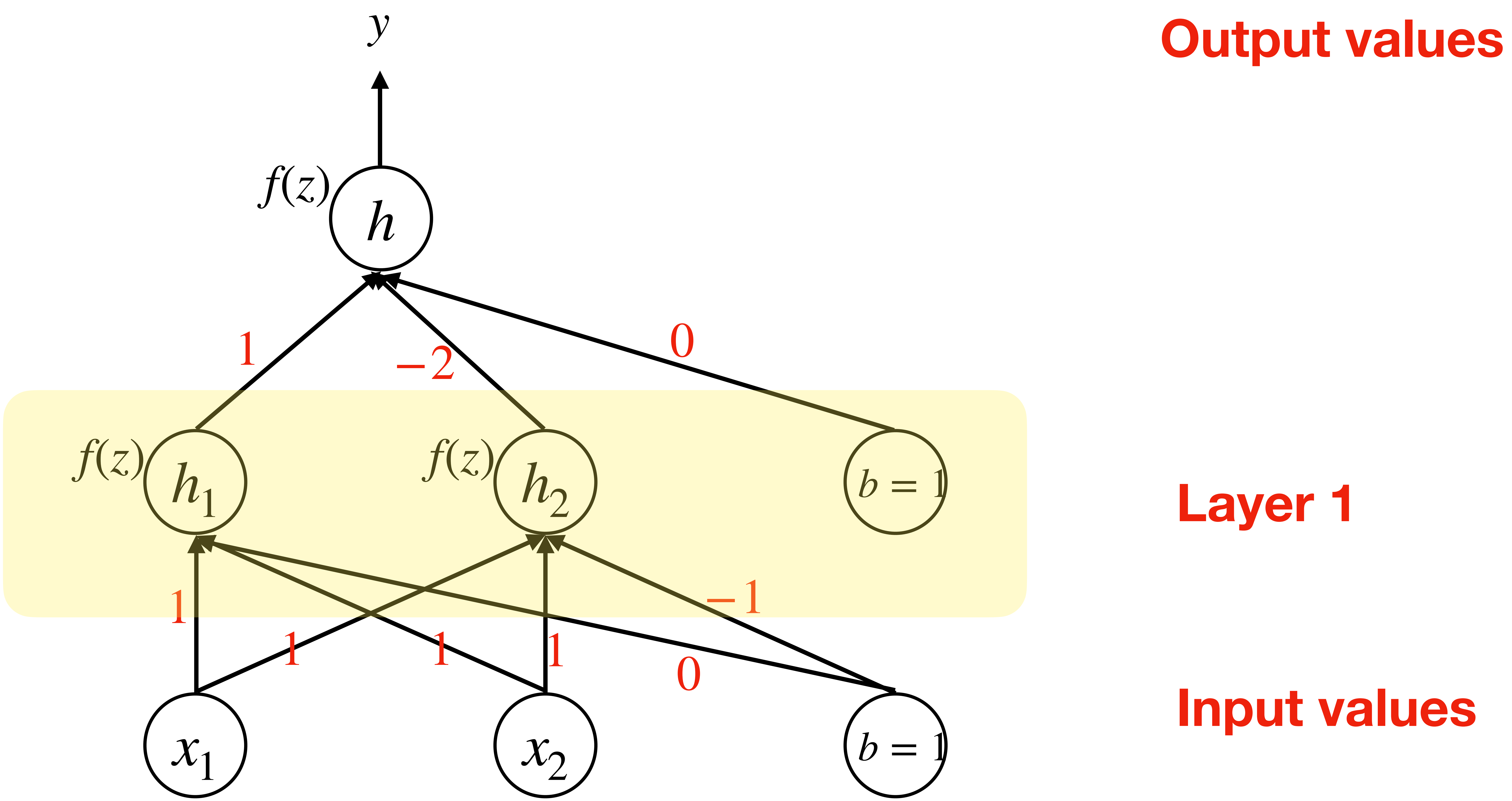

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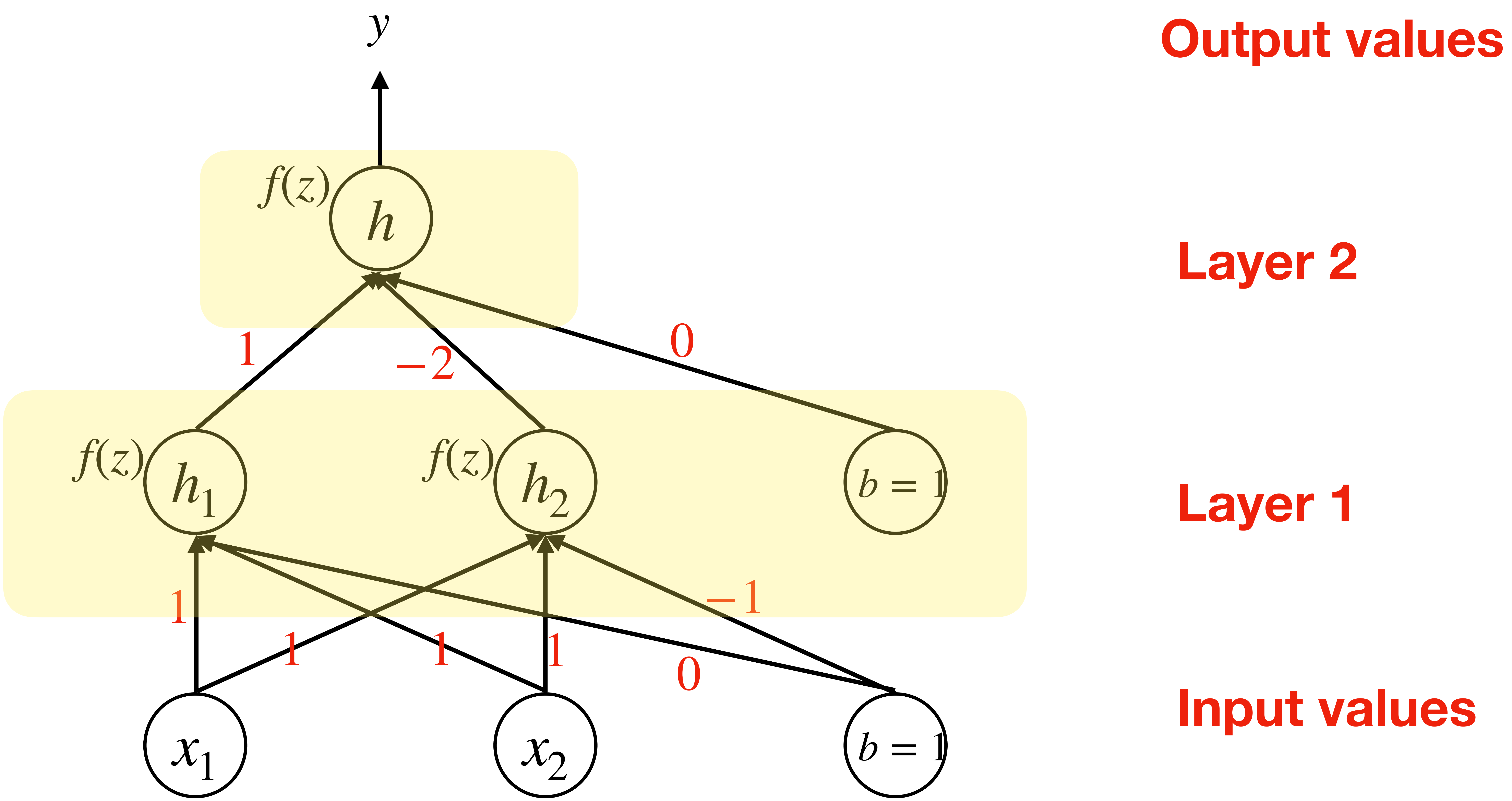
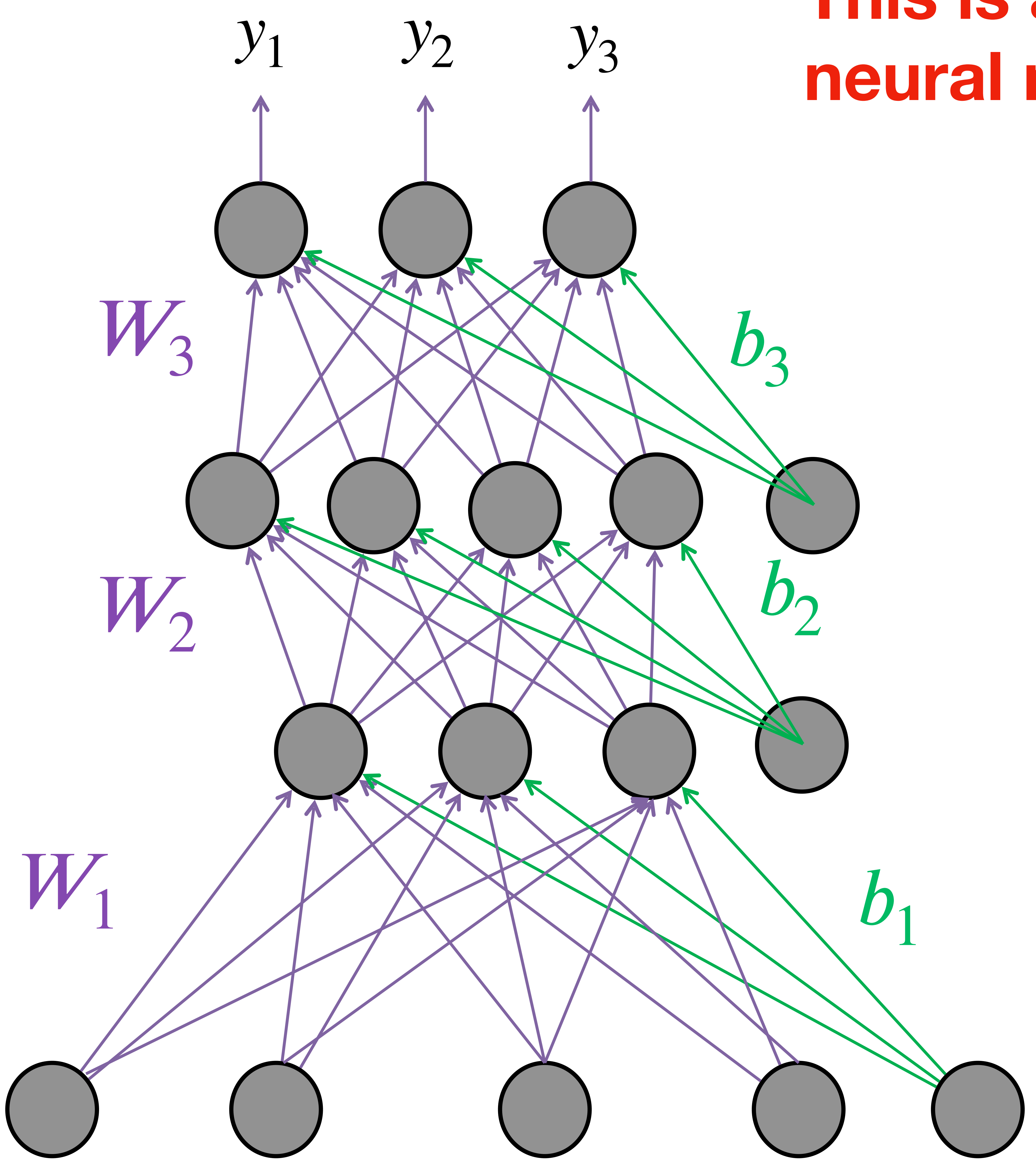


Generalizing the Notations, cont.

Multi-layer Neural Networks

This is a 3-layer neural network



Input layer $\mathbf{x}$ (vector)

Generalizing the Notations, cont.

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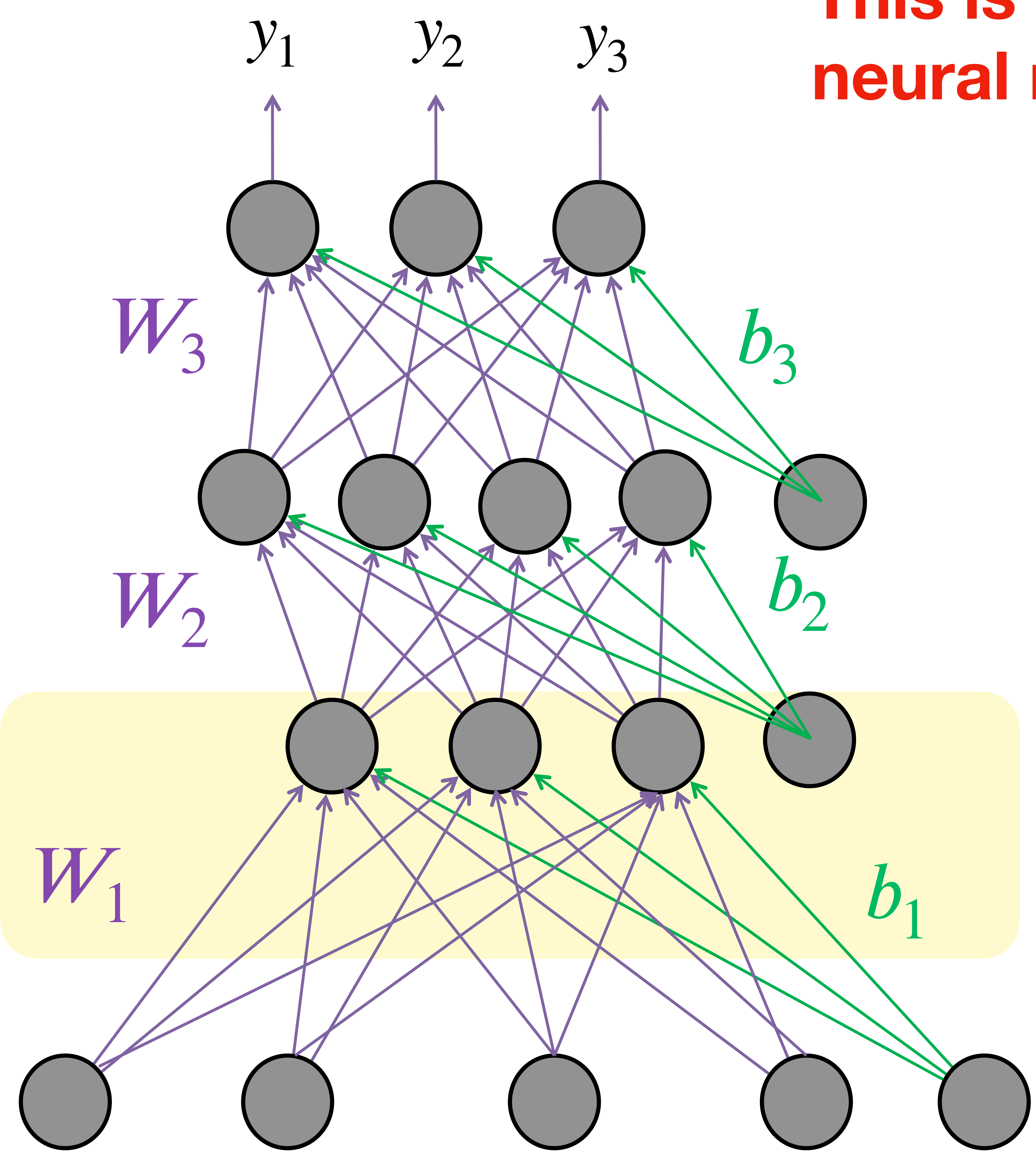

Hidden state $\mathbf{h_1}$ (vector)

$$\mathbf{h_1} = ReLU(\mathbf{W_1 x} + \mathbf{b_1})$$

Input layer $\mathbf{x}$ (vector)

Generalizing the Notations, cont.

Multi-layer Neural Networks

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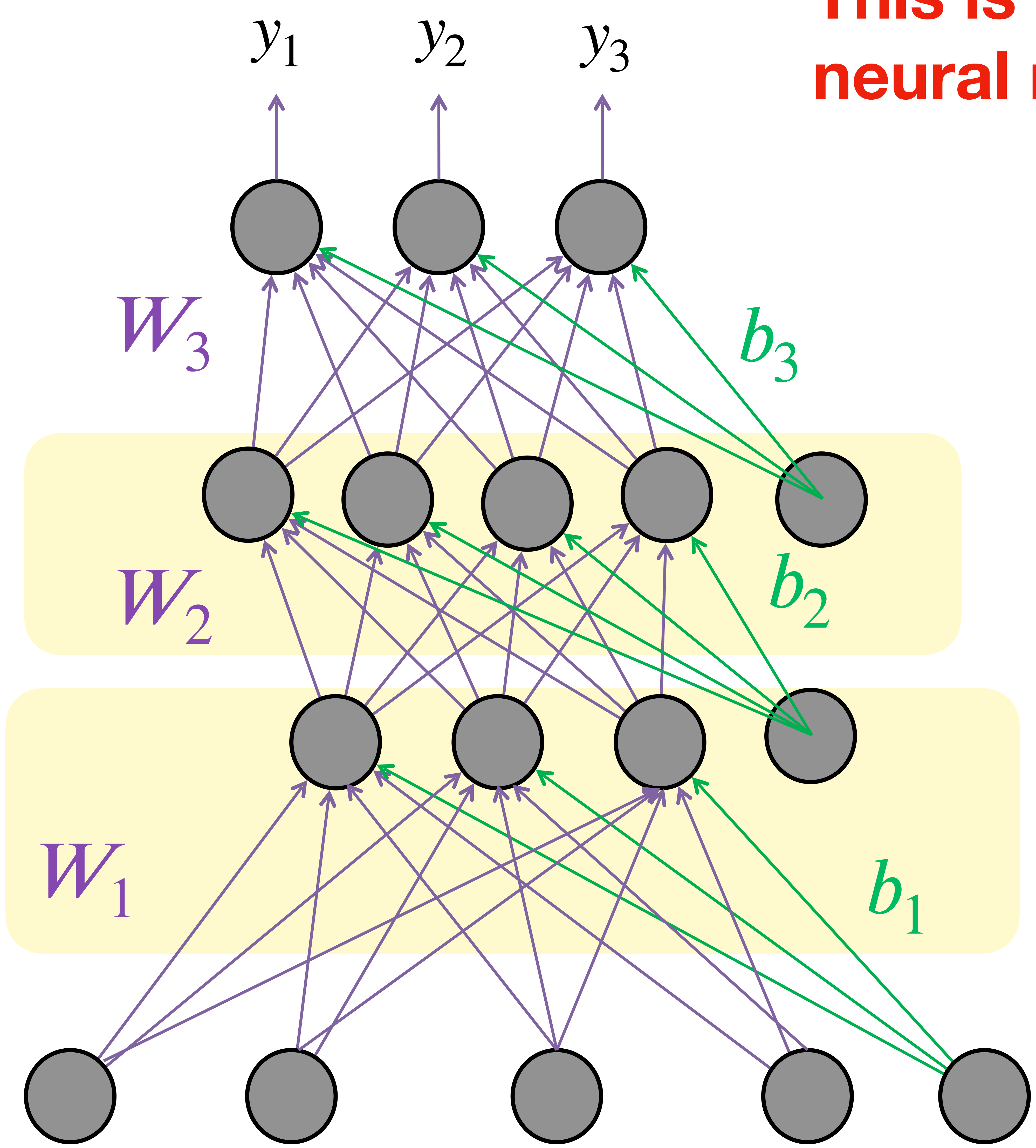

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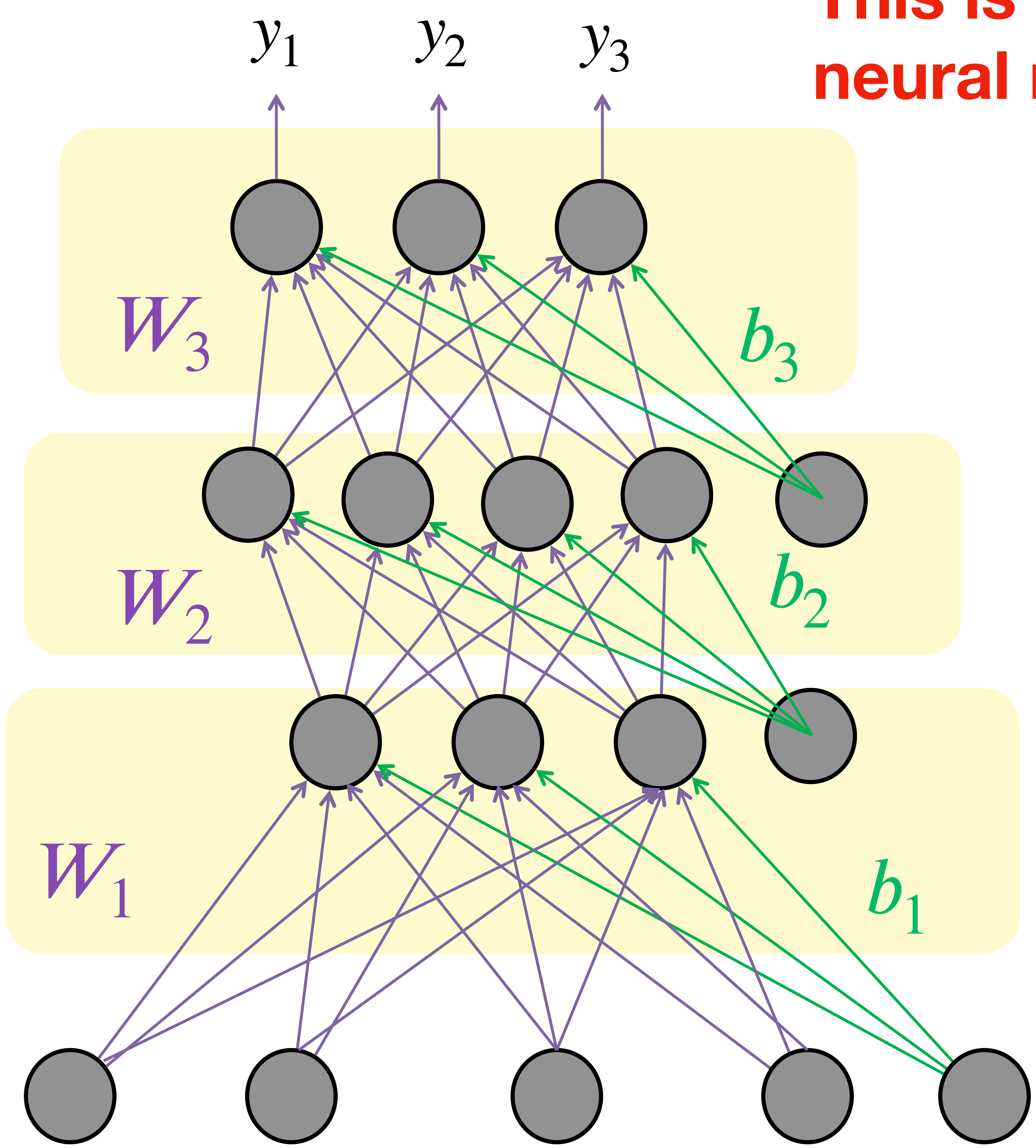

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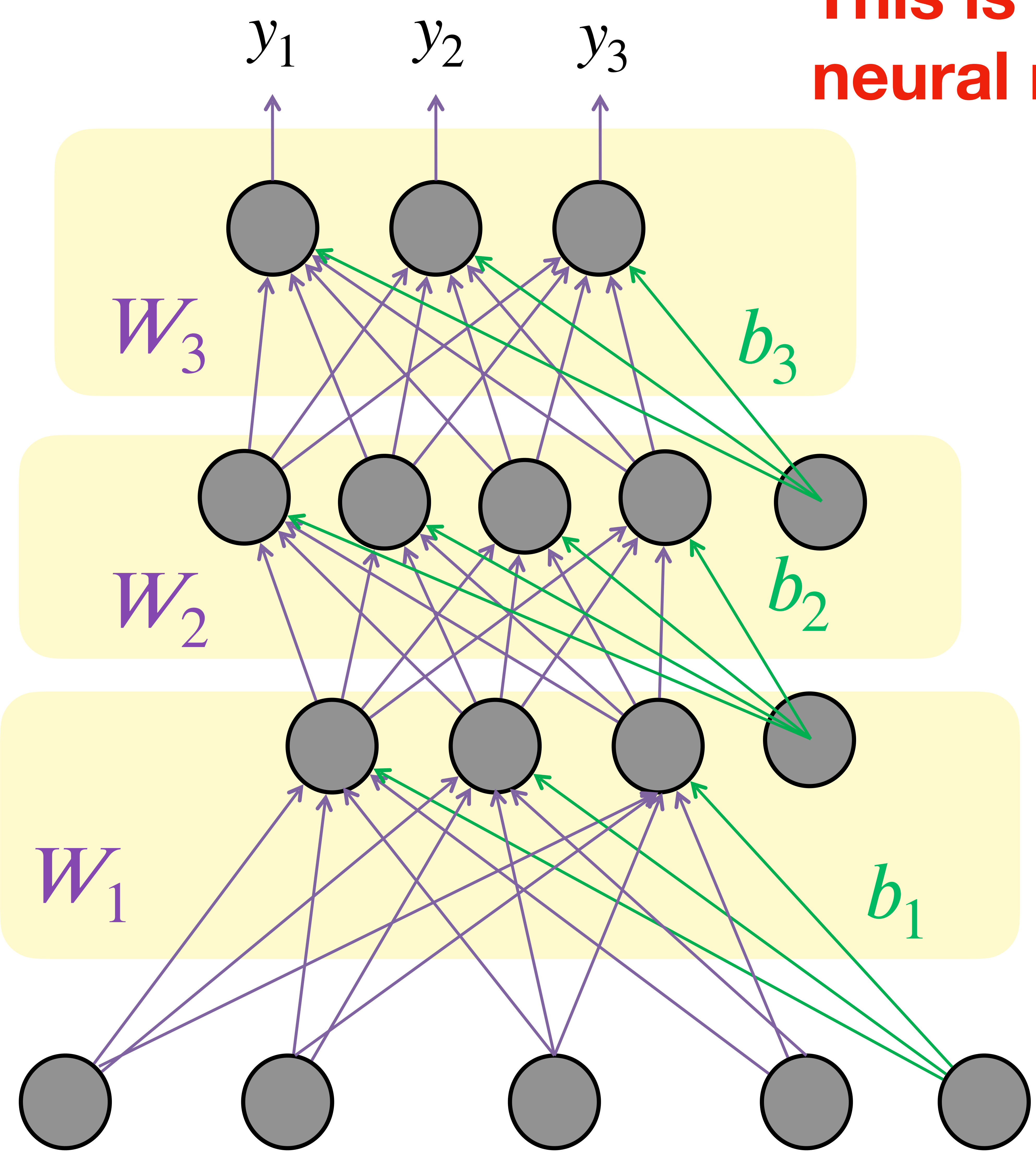

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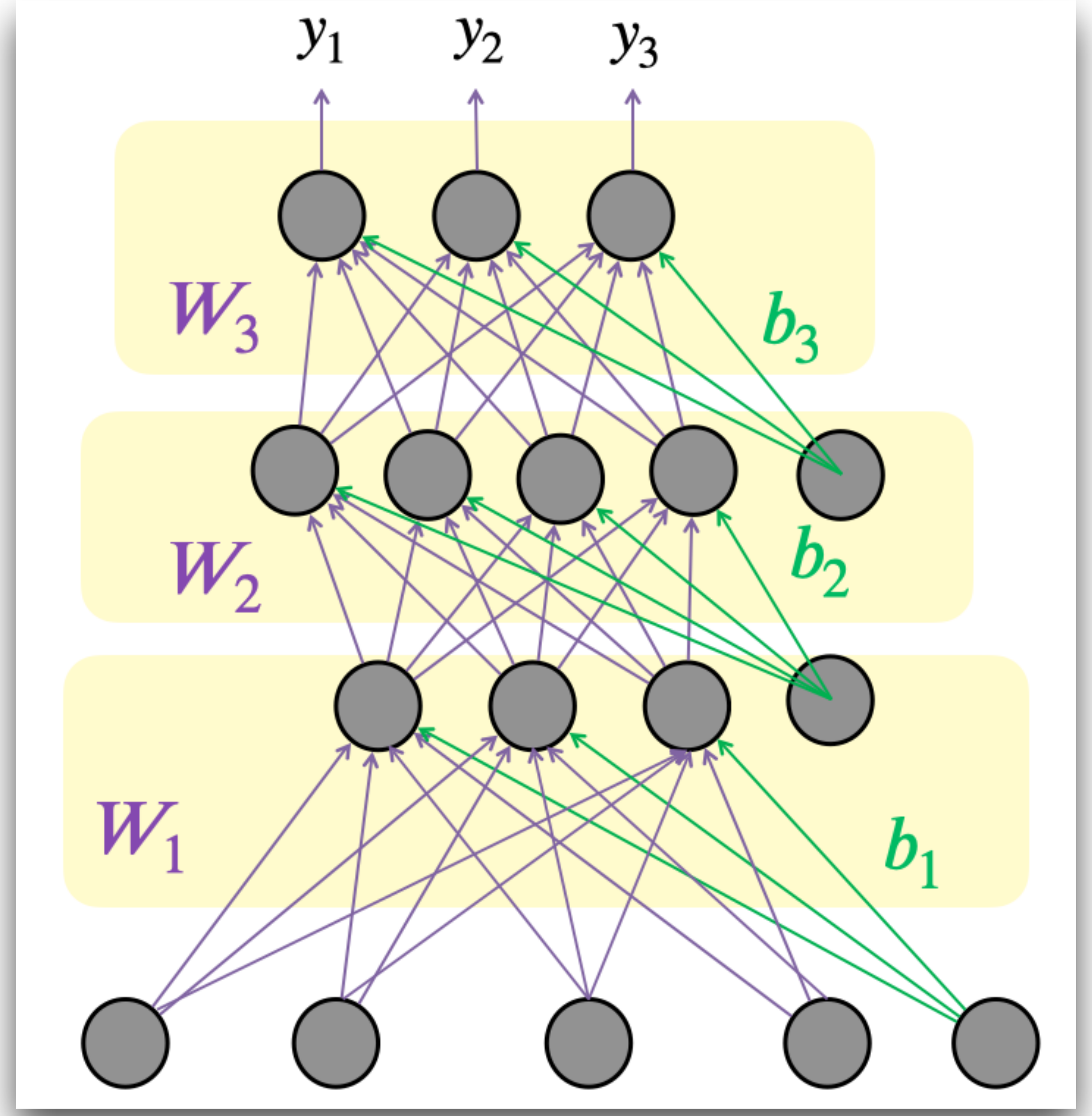
Terminology for this type of Neural Networks

Different names for the same architecture



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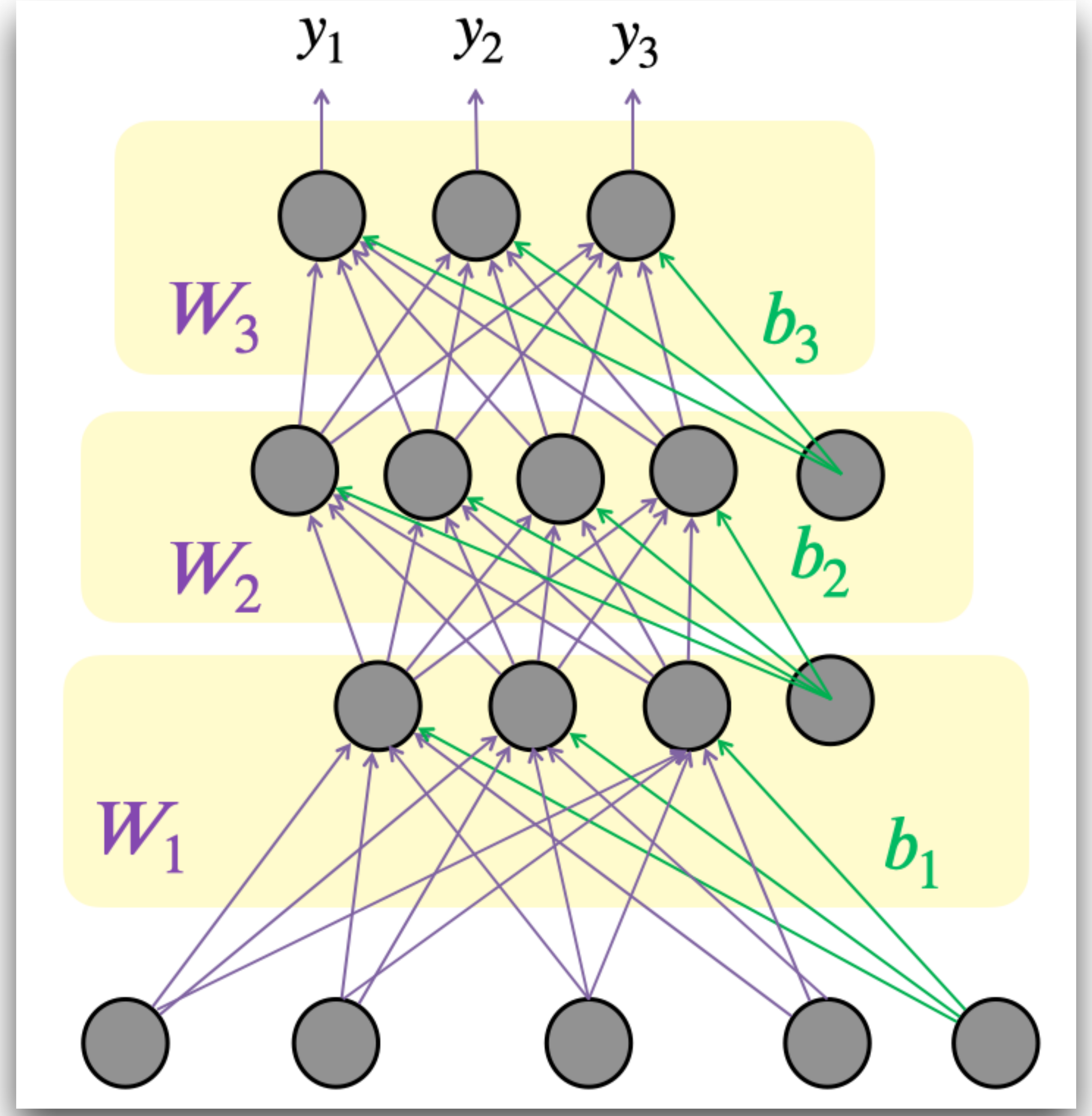
  - ✳ Because layers are feeding foward to each other — the output of layer l becomes the input of layer $l+1$;



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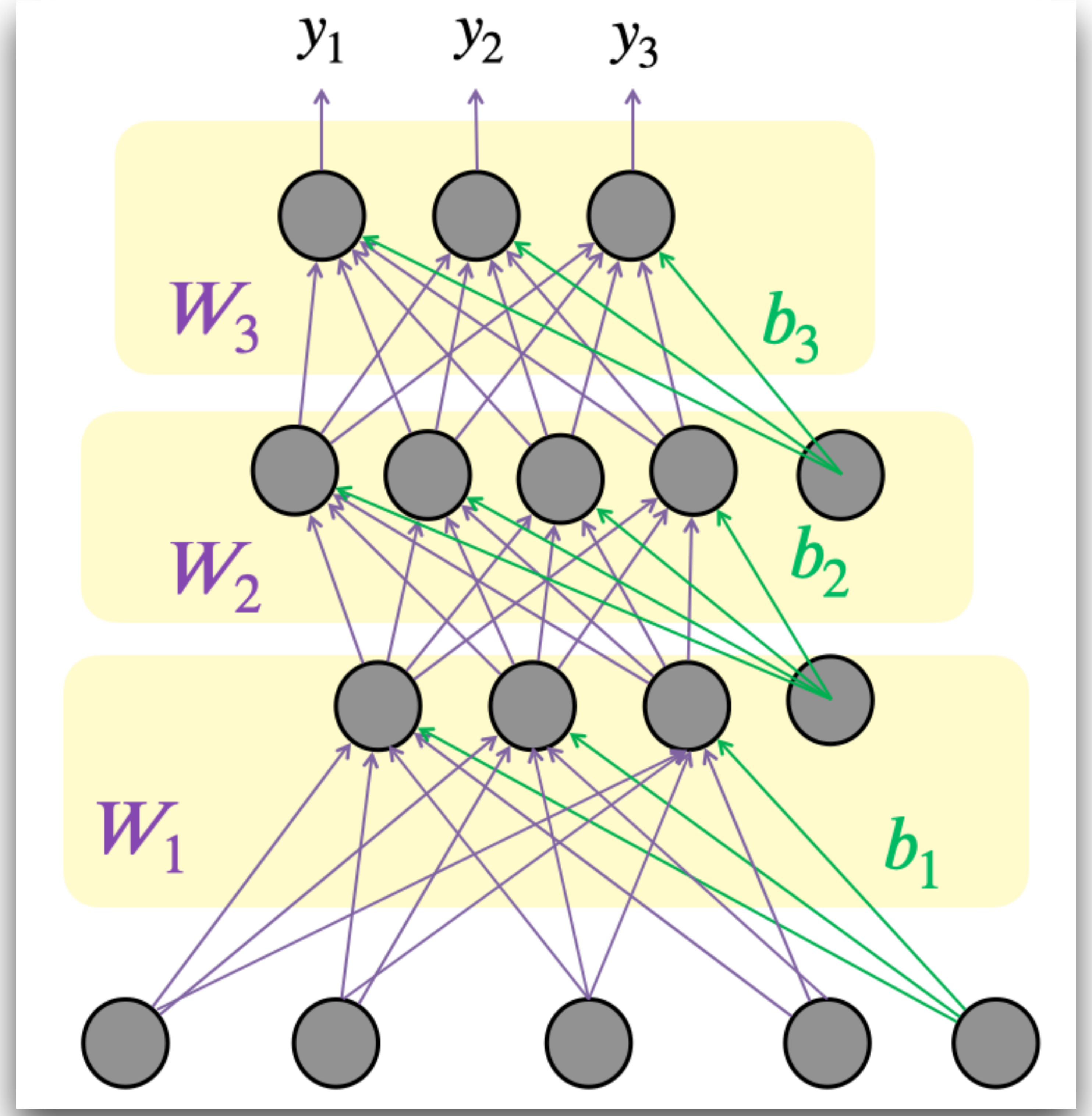

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- **Multi-layer Perceptron (MLP)**
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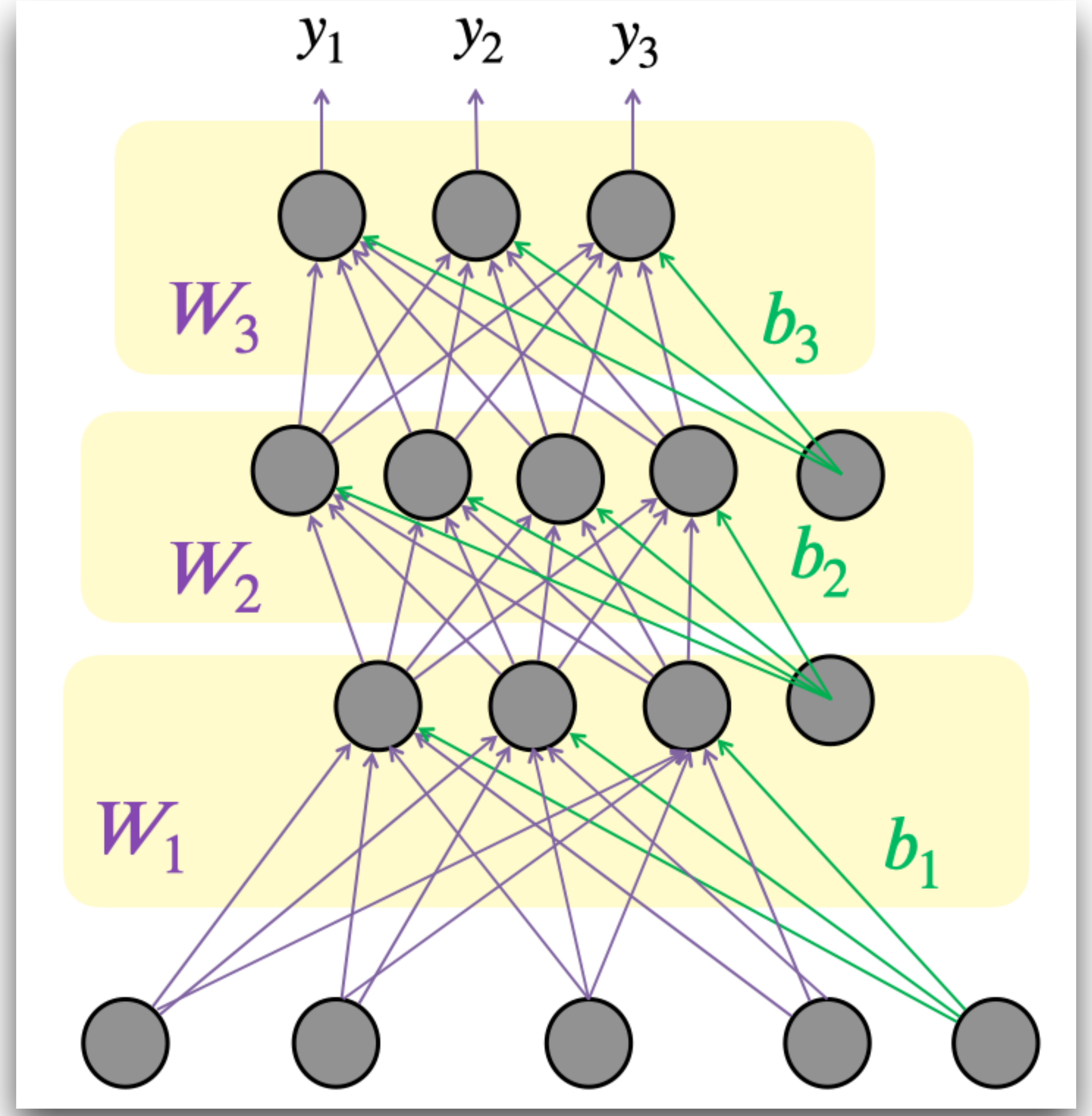

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More generally, at the field level

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- **Deep Learning**
  - ✳ Recent branding for neural networks with many layers;

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A significant upgrade

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- 1-layer Perceptrons are very limited;
- But having one more layer makes it much more powerful!
- In fact, **a 2-layer perceptron can *approximate* any function to an arbitrary degree of precision!**
    - This is an “in principle” claim;
    - Assumes arbitrarily large layers (i.e., infinite neurons);

Multilayer Feedforward Networks are Universal Approximators

KURT HORNIK

Technische Universität Wien

MAXWELL STINCHCOMBE AND HALBERT WHITE

University of California, San Diego

(*Received* 16 *September* 1988; *revised and accepted* 9 *March* 1989)

Interium Summary 3

From individual neurons to networks

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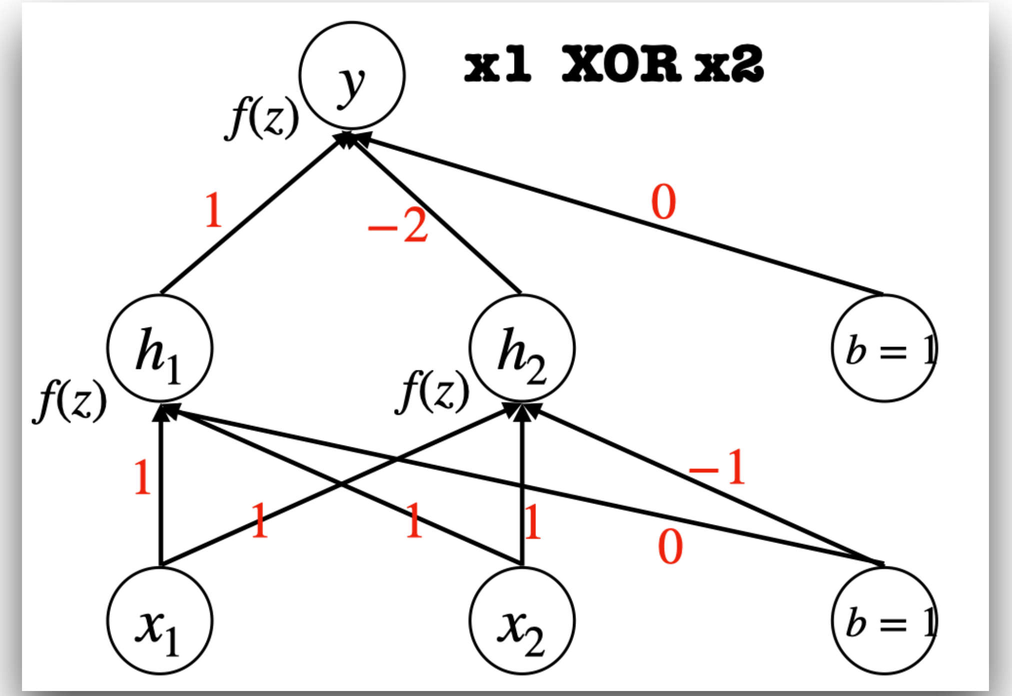

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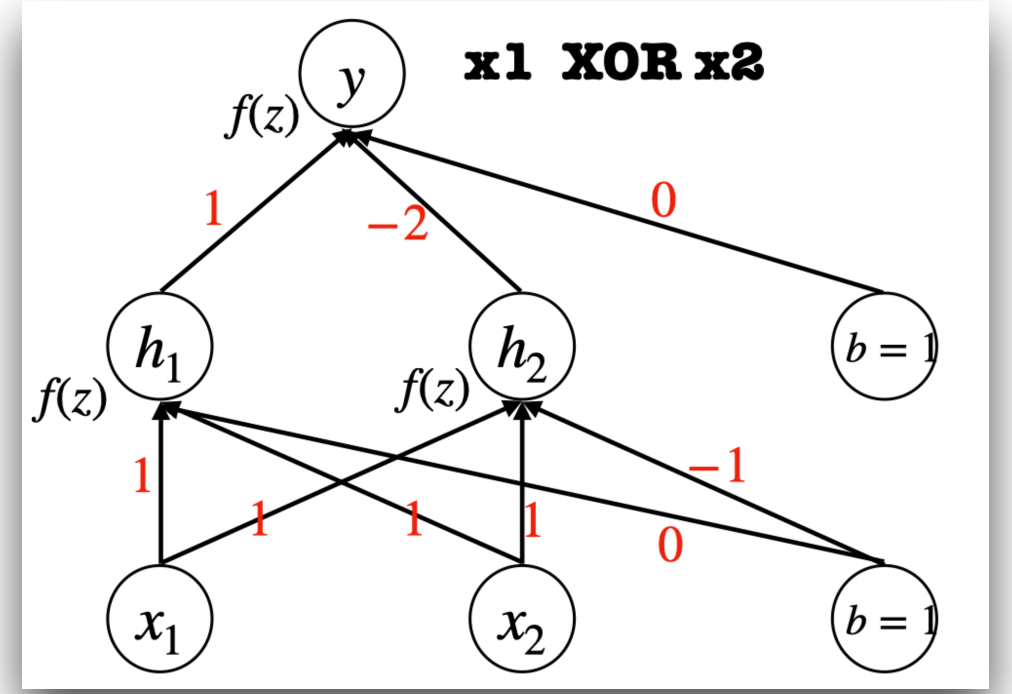

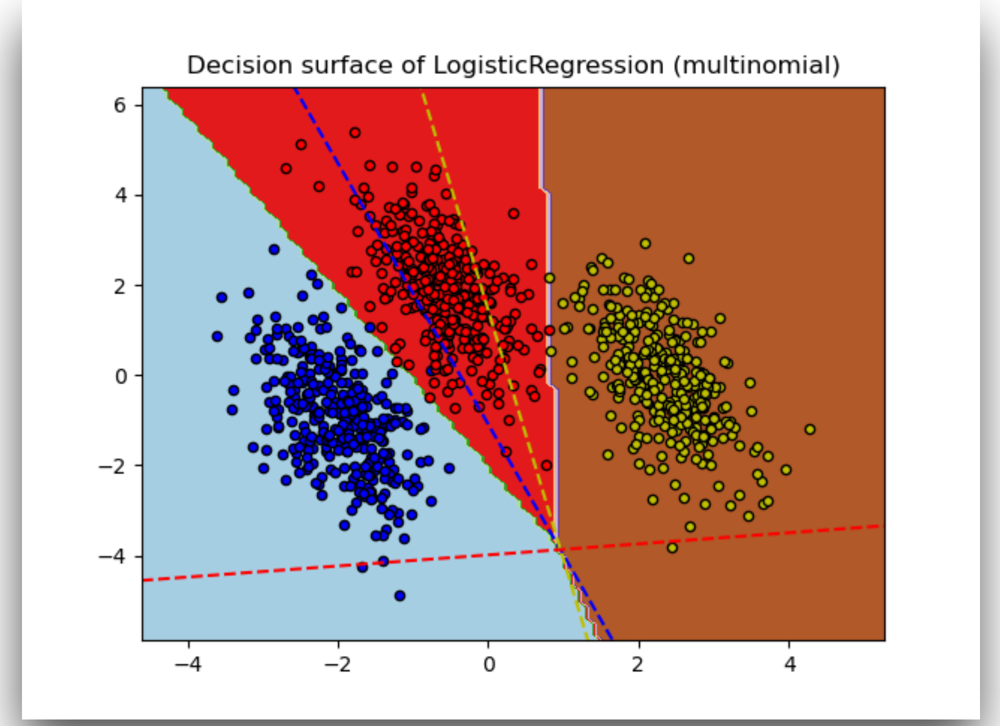

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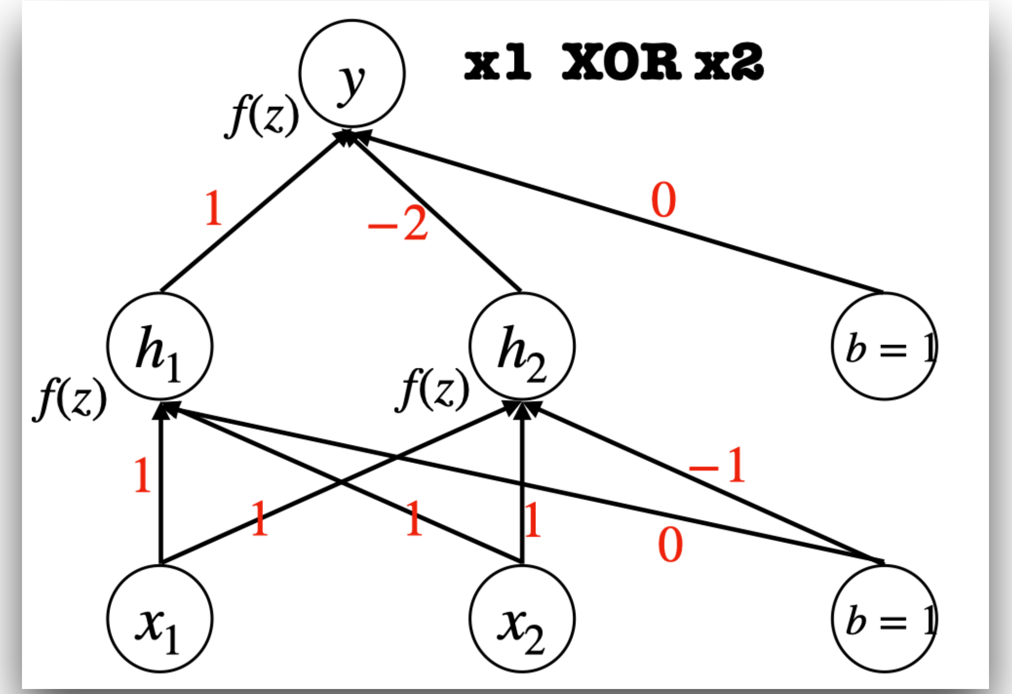

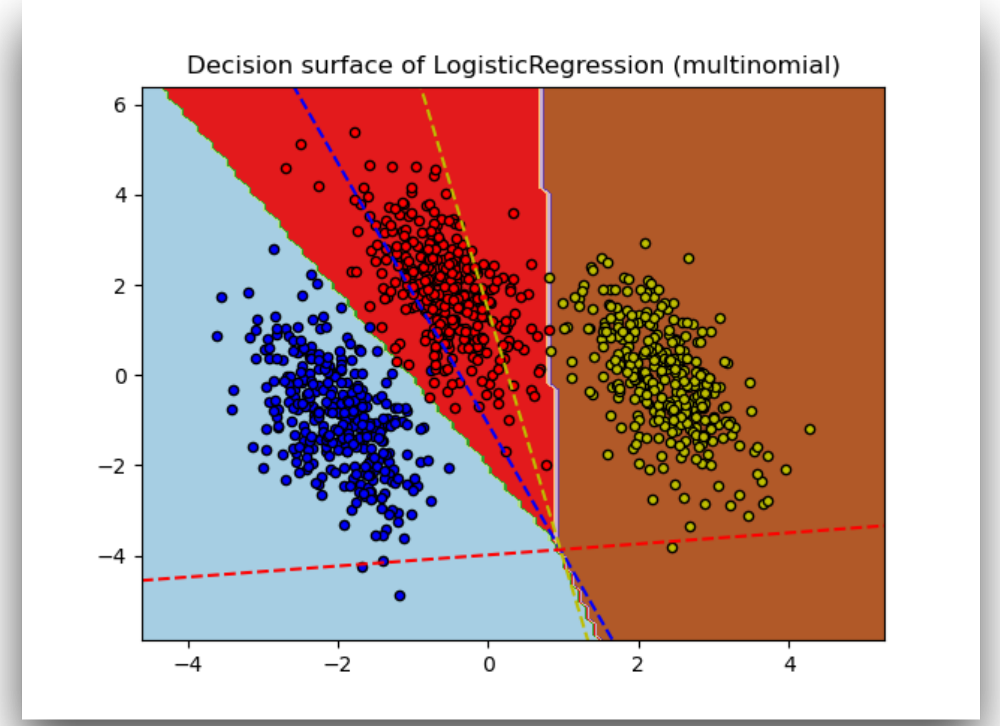

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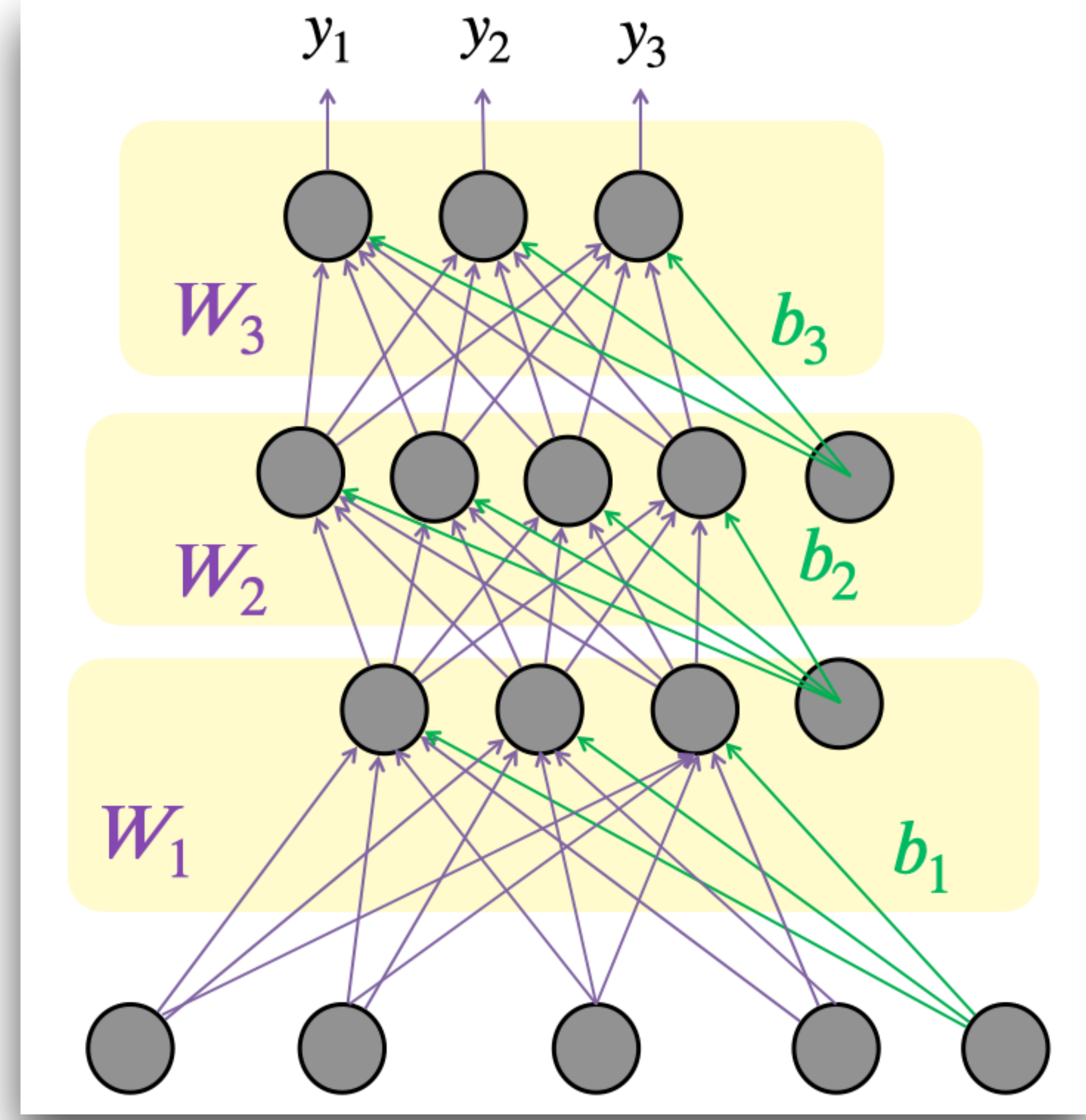

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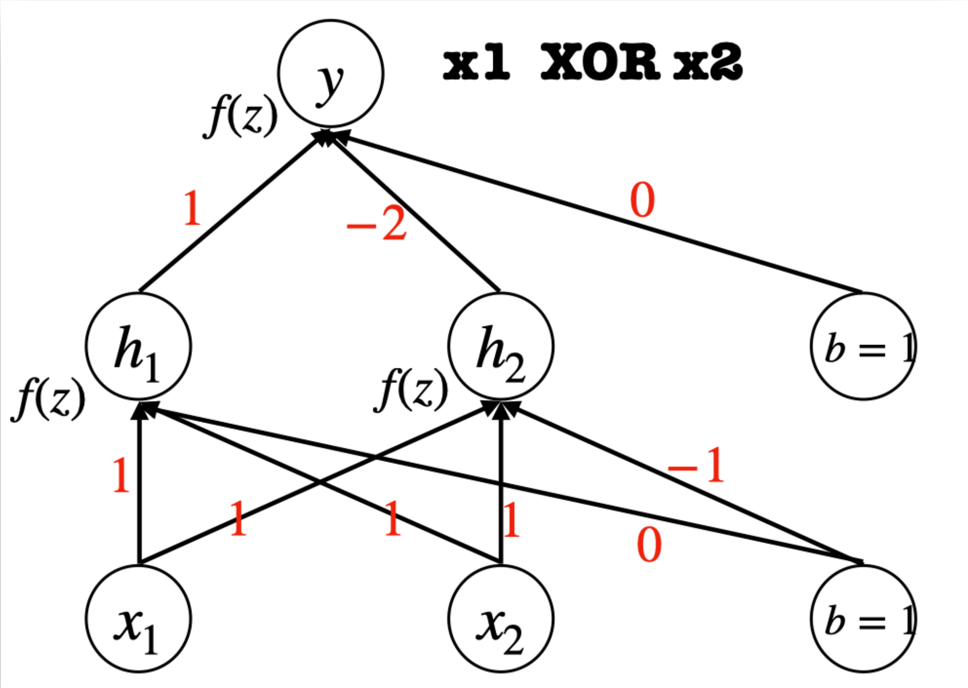

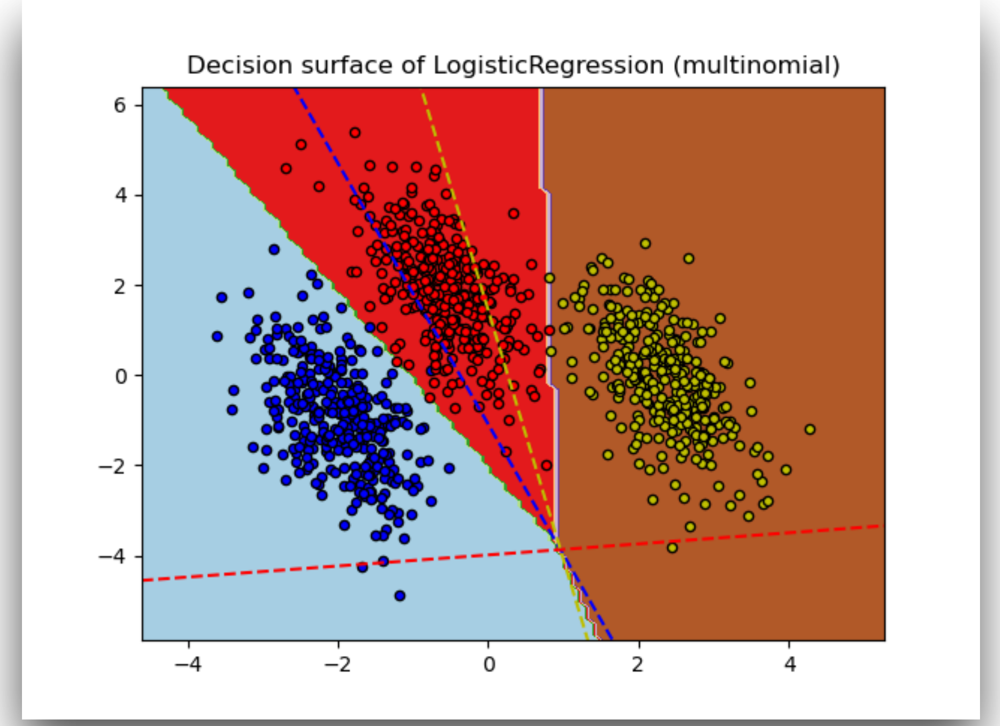

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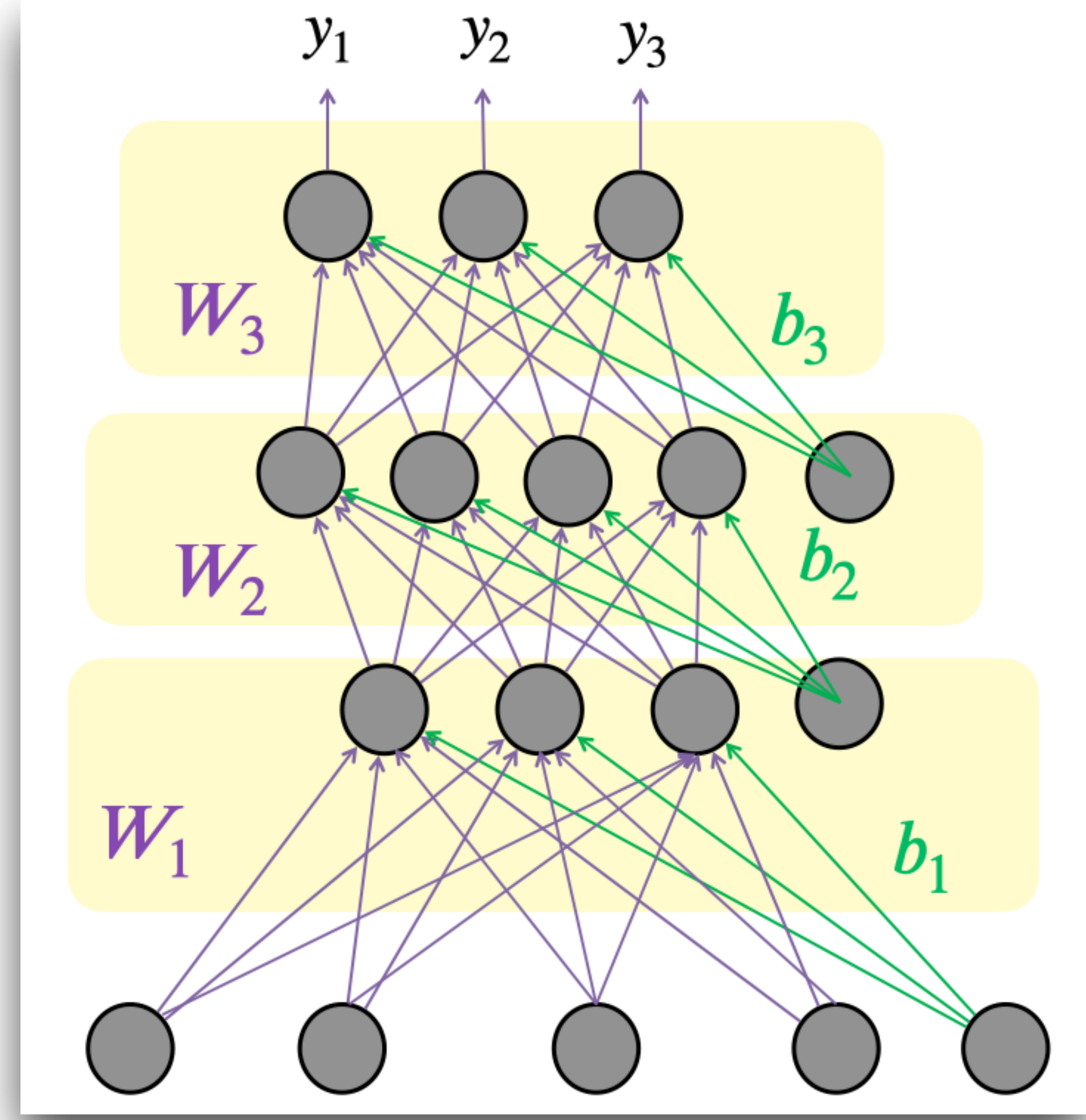


Multilayer Feedforward Networks are Universal Approximators

Kur' Hornik

Technische Universität Wien

Back Propagation & Gradient Descent

1974 (first proposed) & 1986 (polularized)

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Error-based Learning as the secret sauce

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Update each weight: $w' = w - \eta \frac{d}{dw} L(f(x; w), y)$, where $L(f(x; w), y) = L(\hat{y} - y)$

Computation Graphs

Representing the procedure of the computation

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- For training, we need the **derivative of the loss** with respect to **each weight in every layer** of the network;
  - But the loss is computed only at the very end of the network….
  - How do we know how much a weight at Layer 1 contribute the final loss?

Computation Graphs

Representing the procedure of the computation

- For training, we need the **derivative of the loss** with respect to **each weight in every layer** of the network;
    - But the loss is computed only at the very end of the network….
    - How do we know how much a weight at Layer 1 contribute the final loss?
- **Solution** = represent the computation of the entire neural network with a (very big) *computation graph*!
    - A computation graph represents the entire process of computing a function (which can be a very complex composition of multiple functions) — representing the dependencies between any two steps of the computation.

Computation Graphs

[Exercise] Try to construct a computation graph for a simple equation

- Say, here is a dummy loss function with 3 inputs: $\underline{L(a,b,c) = c(a+2b)}$
- In a computation graph: a **node** represents a value, and an **edge** represents a function / arithmetic operation / computation.
- Try to construct a computation graph for L!

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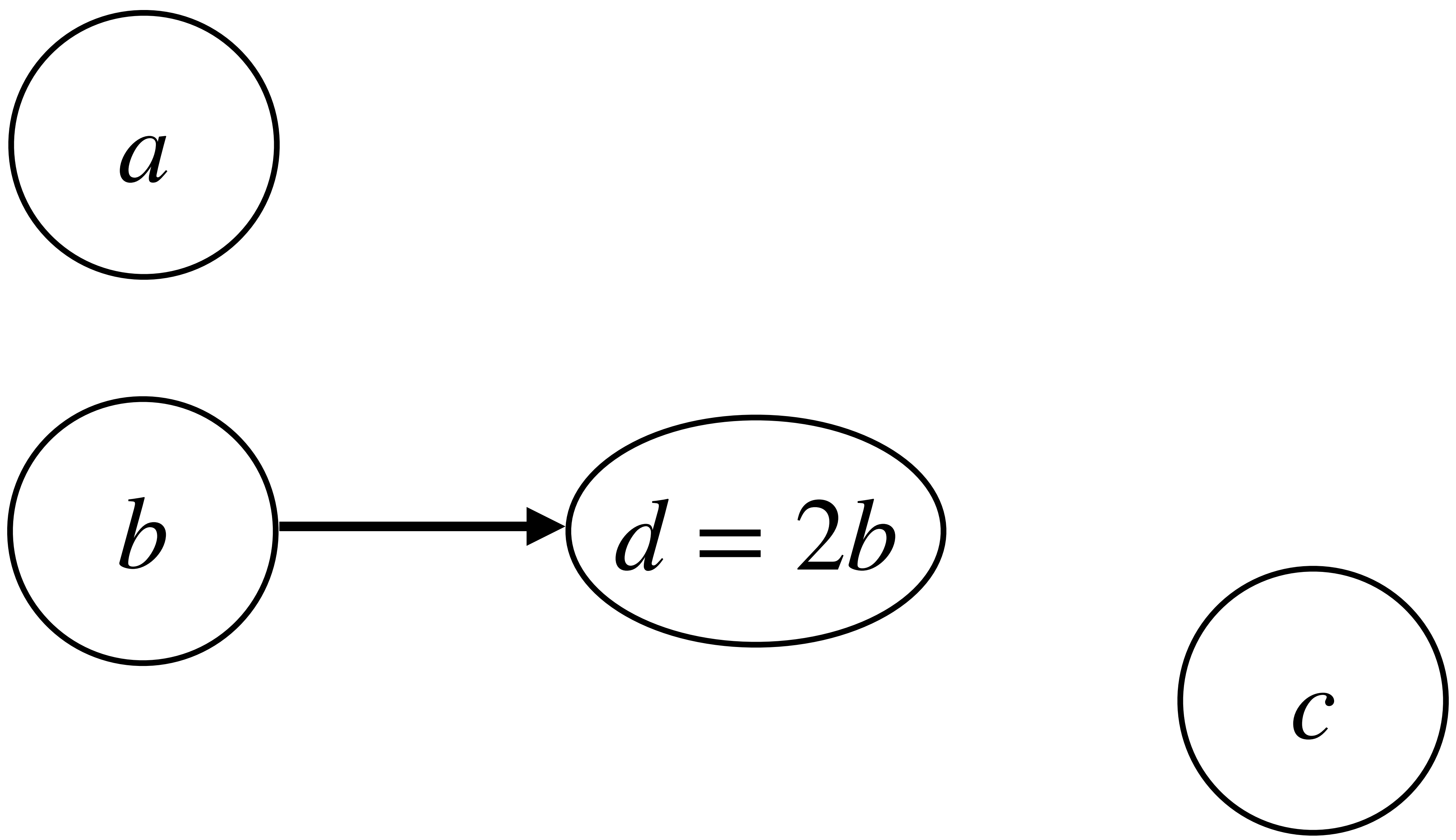

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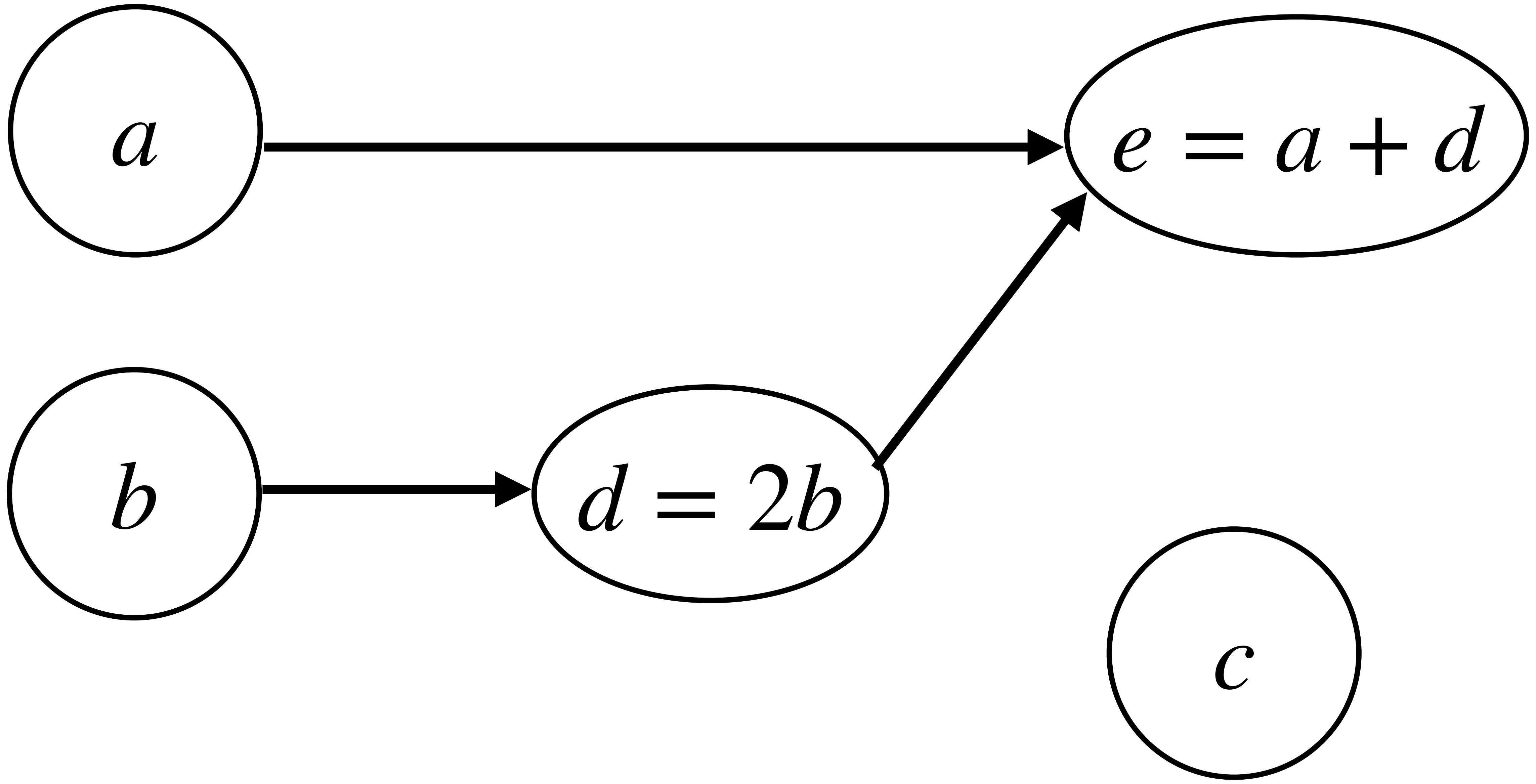

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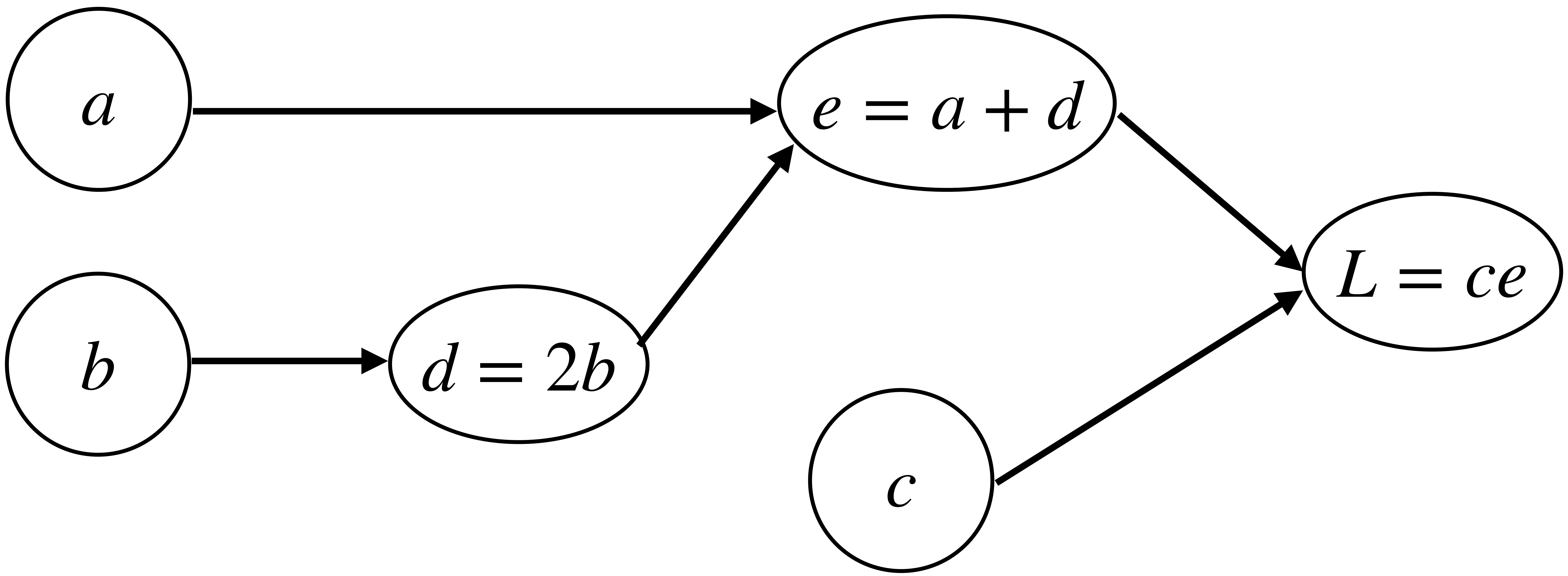
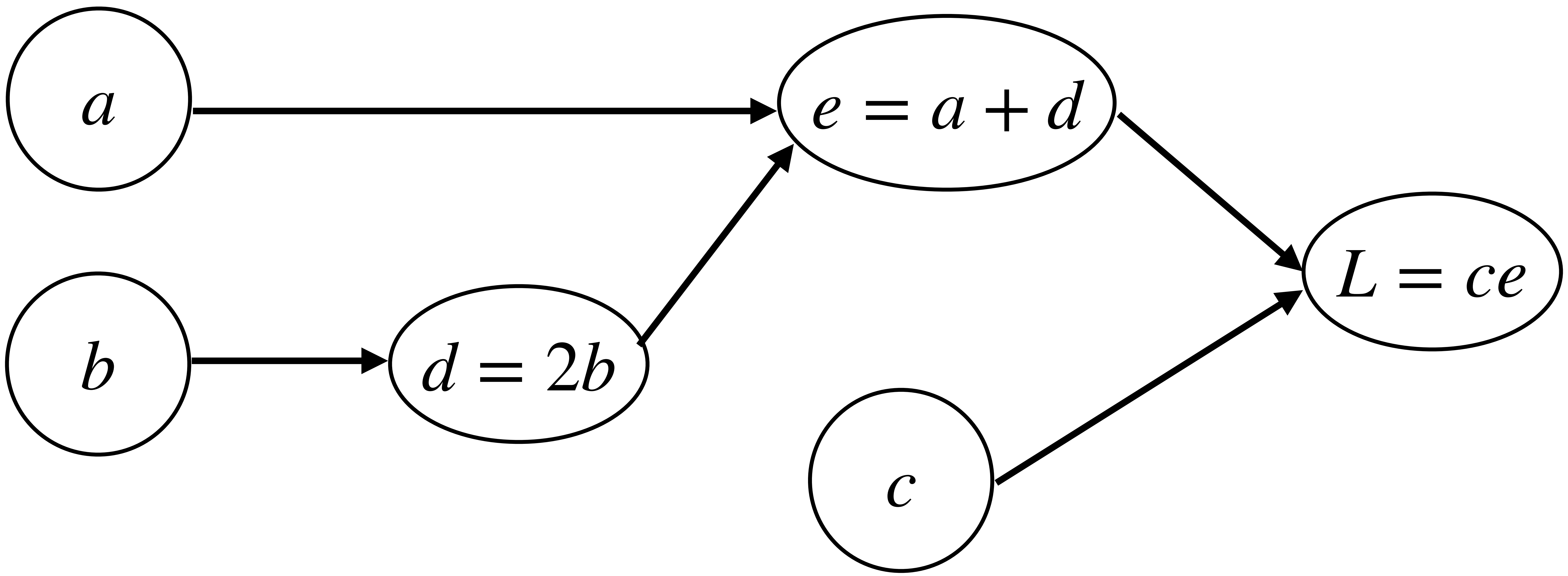


Computation Graphs

[Exercise] Try to construct a computation graph for a simple equation

- $L(a, b, c) = c(a + 2b)$. Now let $a = 3, b = 1, c = -2$



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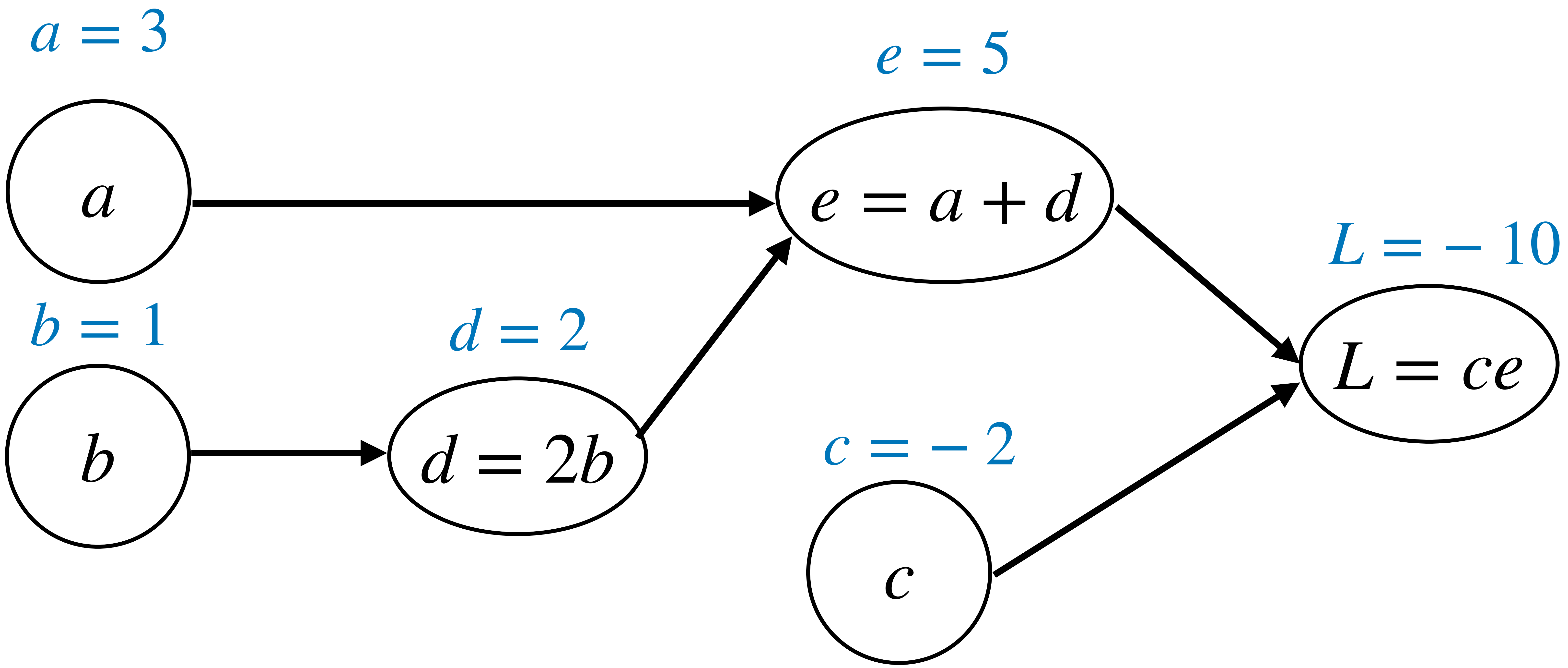

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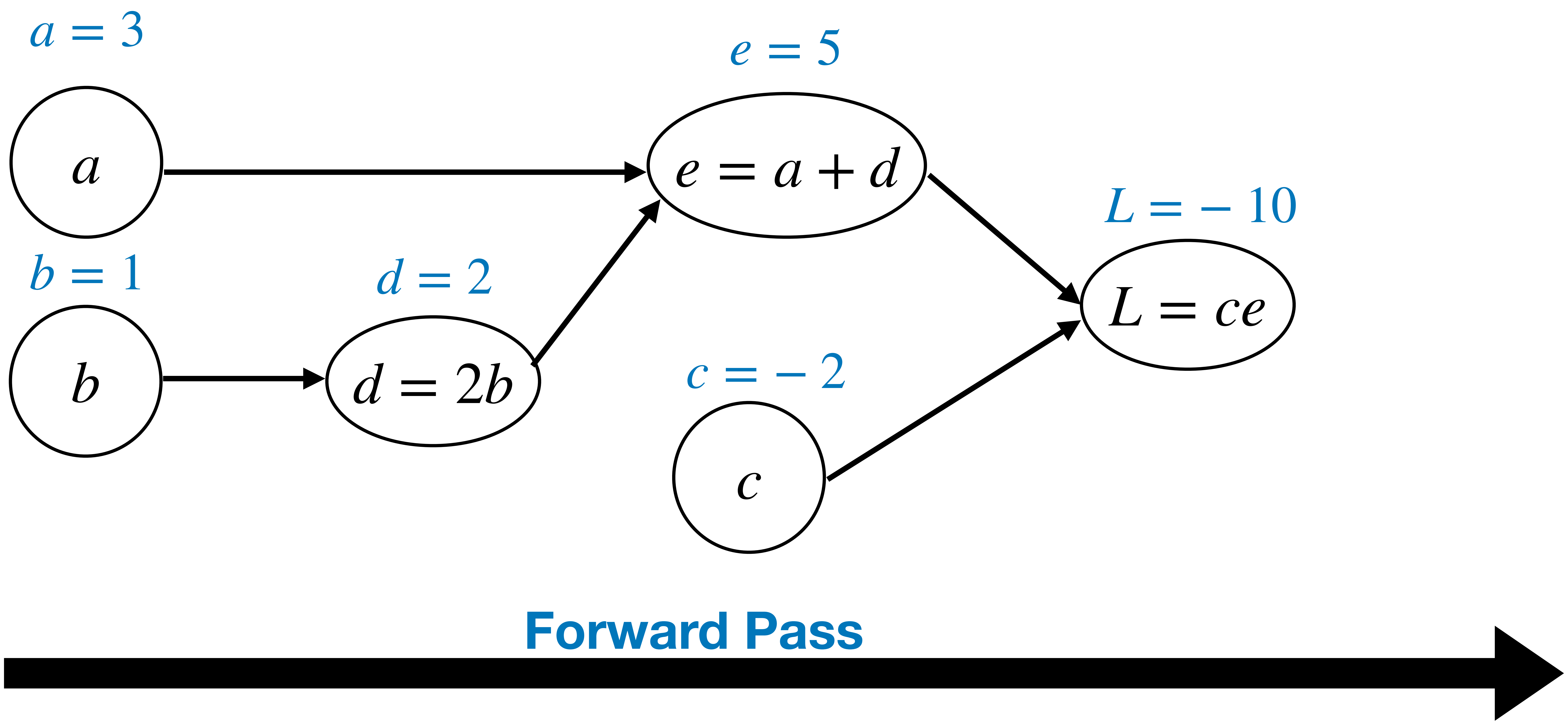

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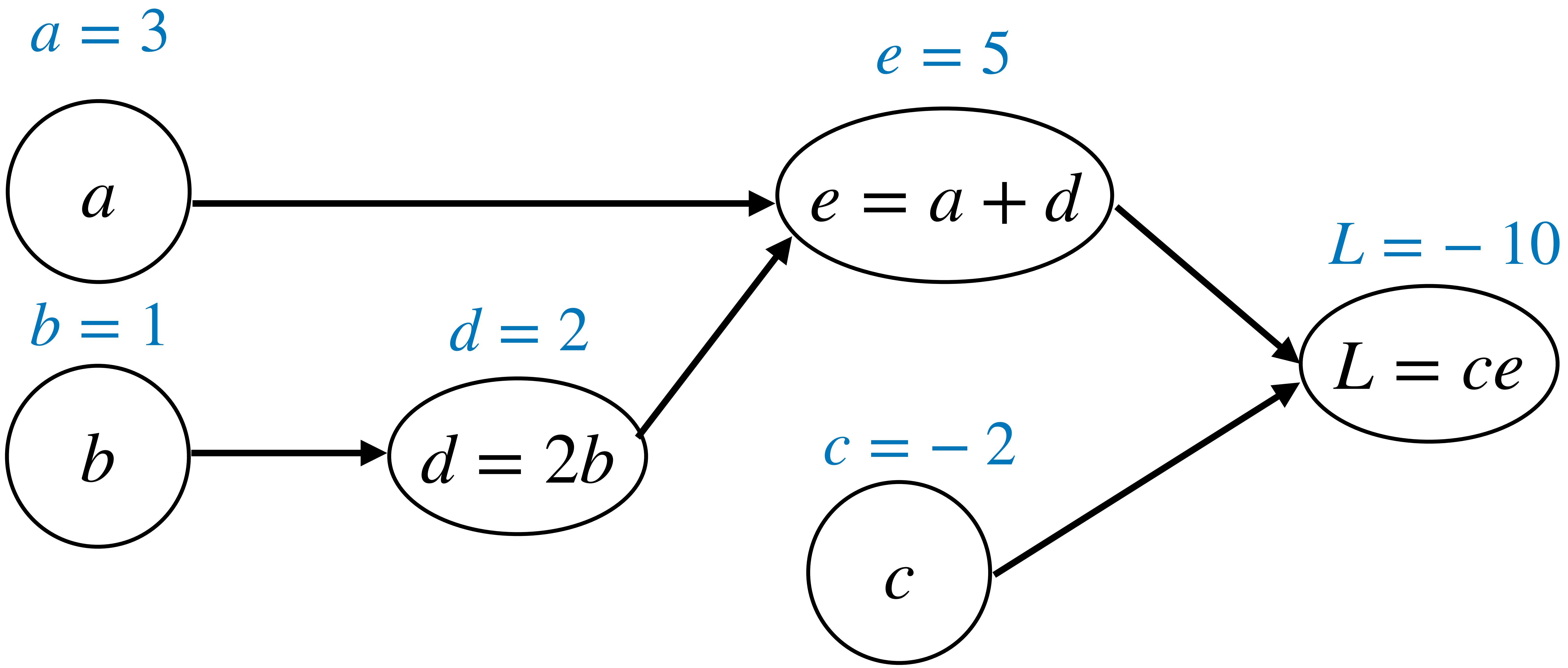
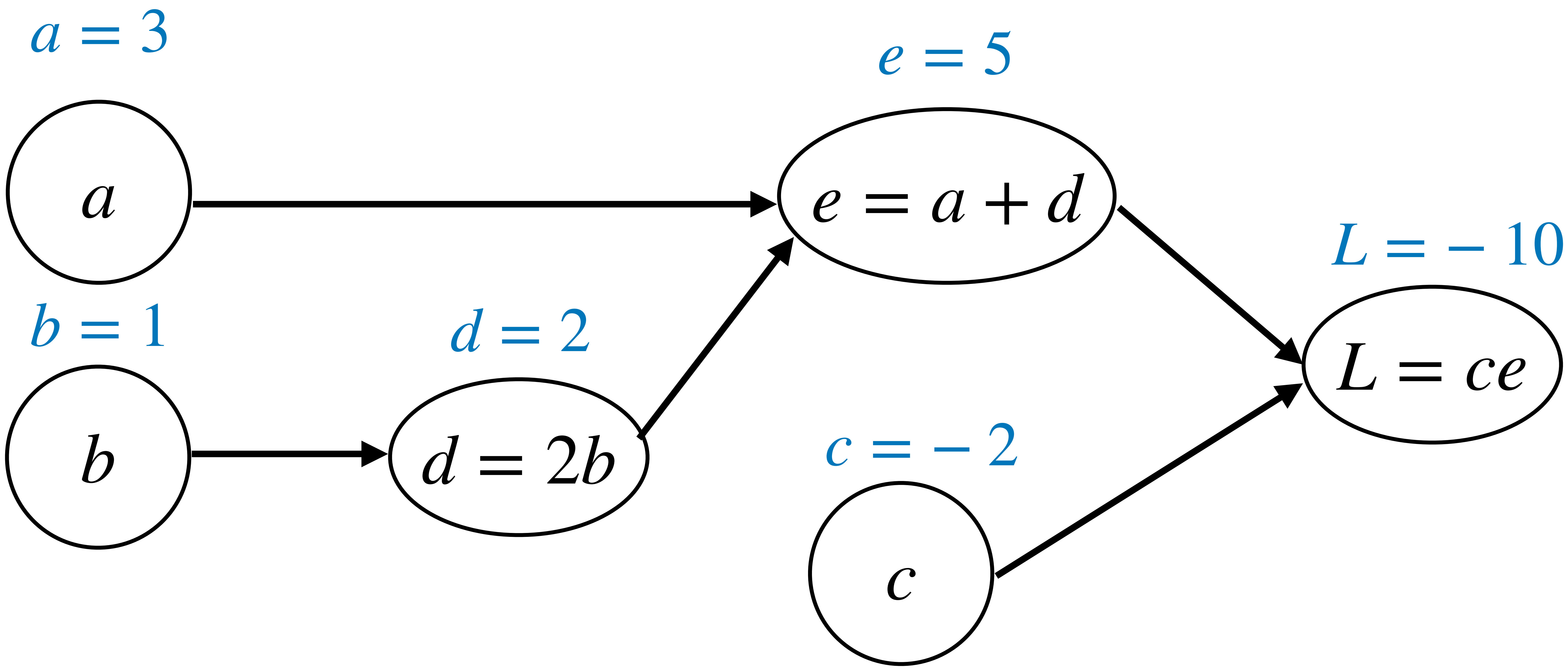


Remember:
We want to know how much does a, b, c each contribute to the final loss L

Forward Pass

Backpropagation

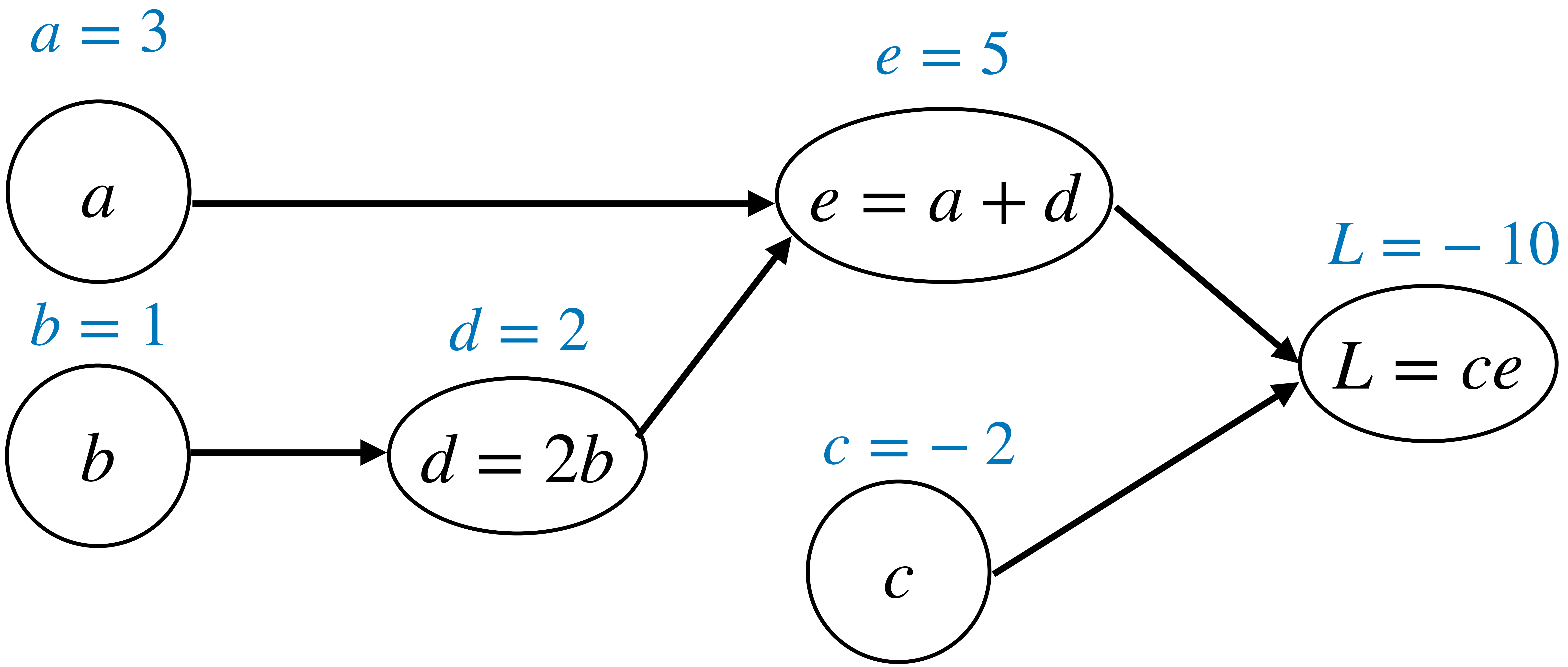
Responsibility attribution through gradient



Backpropagation

Responsibility attribution through gradient

- $\underline{L(a,b,c) = c(a+2b)}$. Now let $a = 3, b = 1, c = -2$;
- We don't know immediately how much a contribute to L…
- But we do know how much c and e contributes (since they are the final step)!



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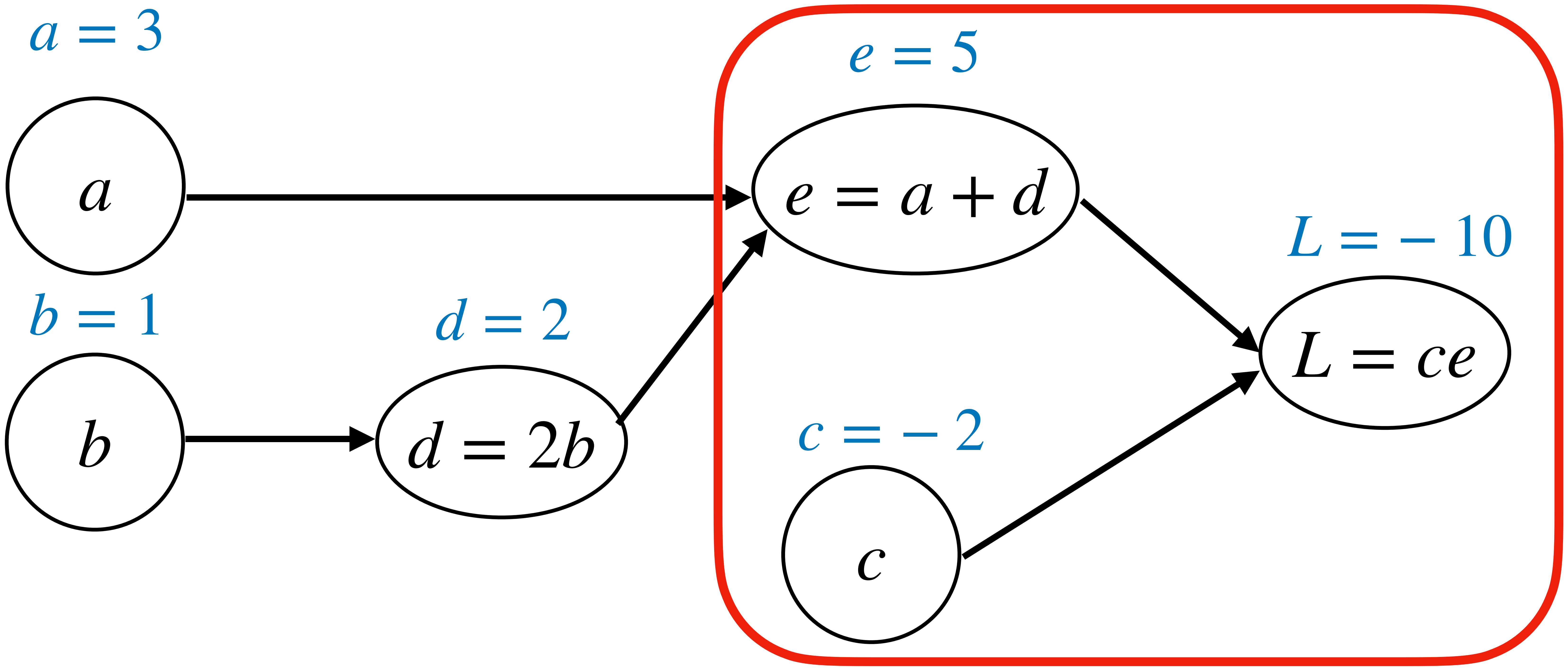

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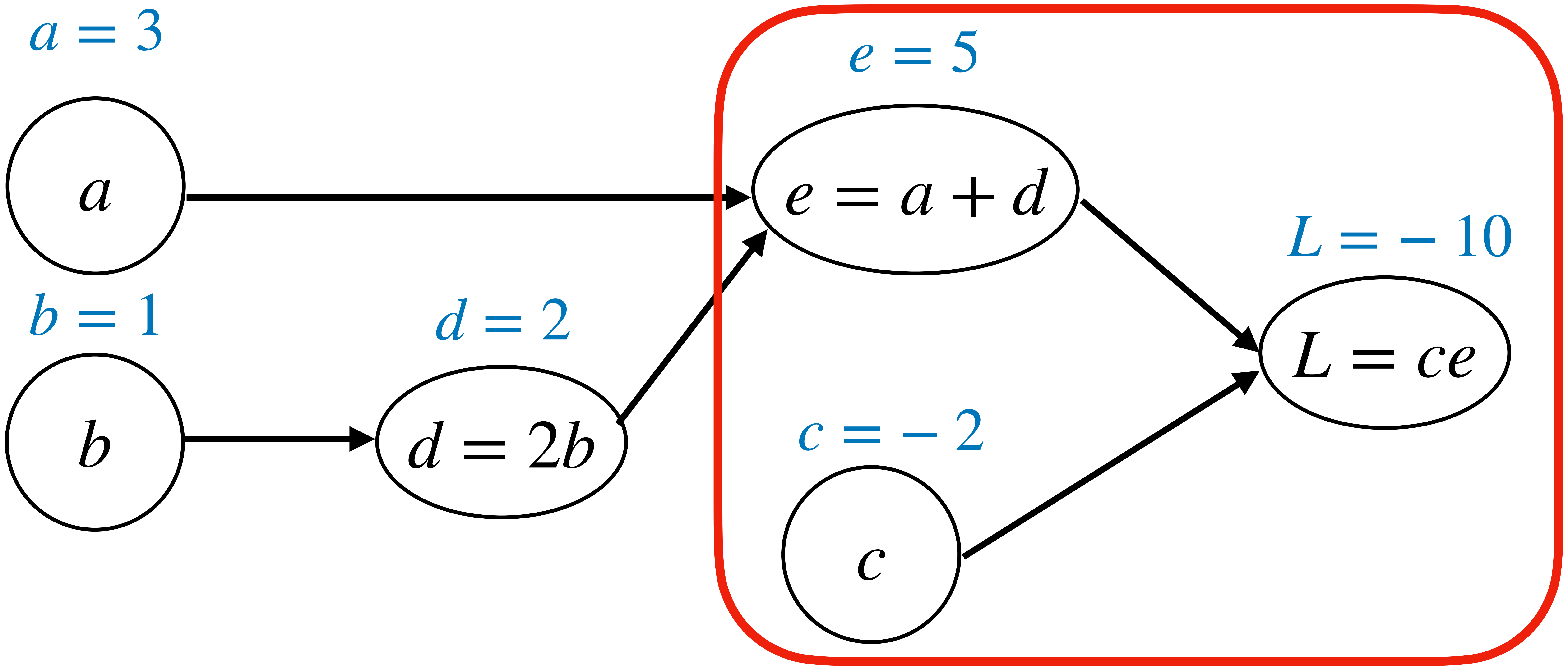

$L = ce$

Backpropagation

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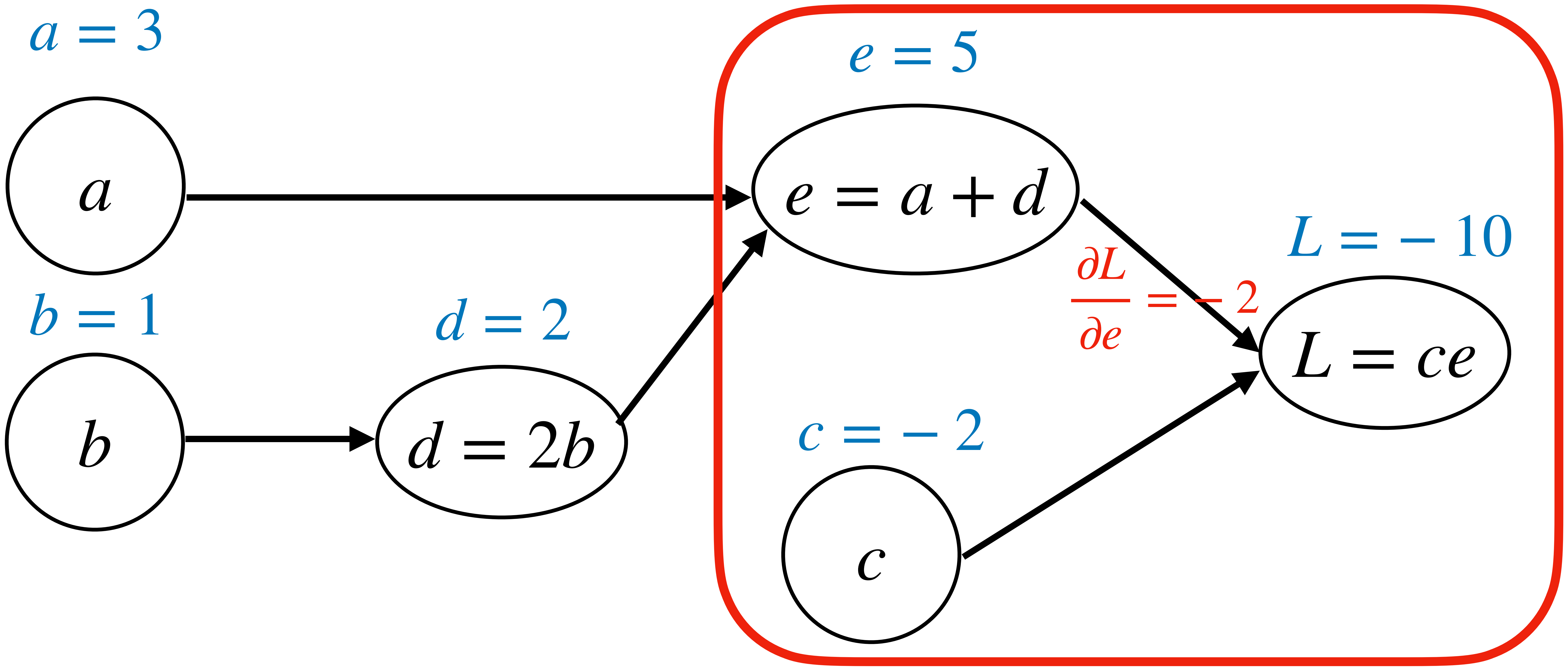

$L = ce$

$\Rightarrow \frac{\partial L}{\partial e} = c = -2$

Backpropagation

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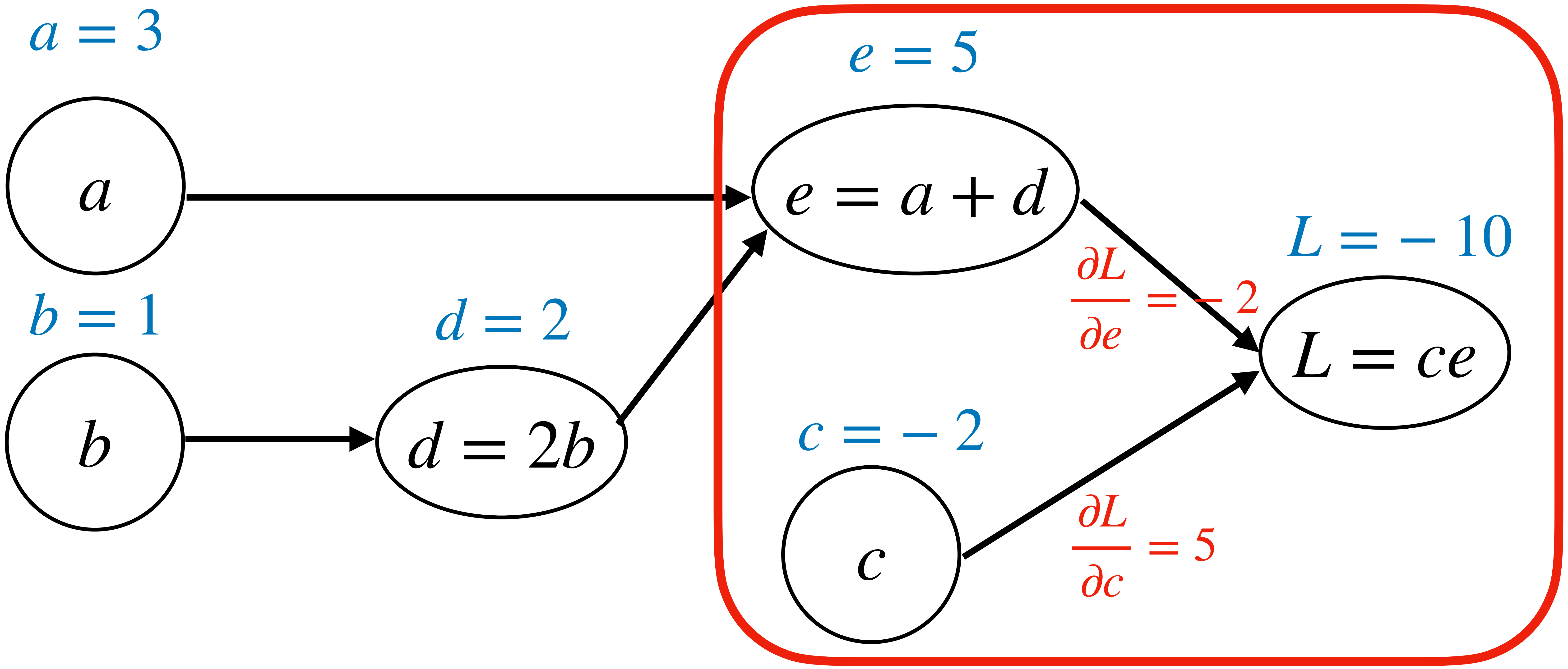


$$L = ce$$

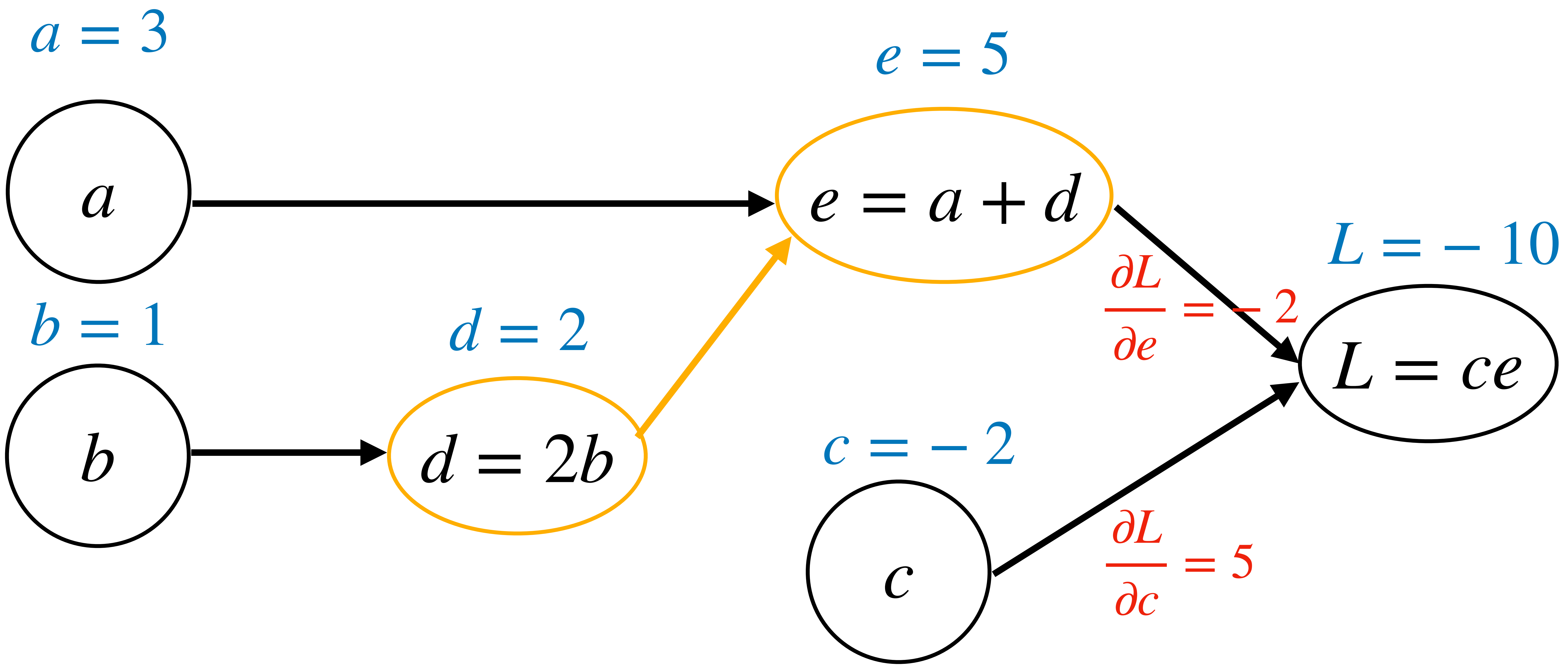
$$\Rightarrow \frac{\partial L}{\partial e} = c = -2$$

$$\Rightarrow \frac{\partial L}{\partial c} = e = 5$$

Backpropagation

Responsibility attribution through gradient

- $\underline{L(a, b, c) = c(a + 2b)}$. Now let $a = 3, b = 1, c = -2$;
- We don't know immediately how much a contribute to L…
- We use e's contribute to compute d's contribution.



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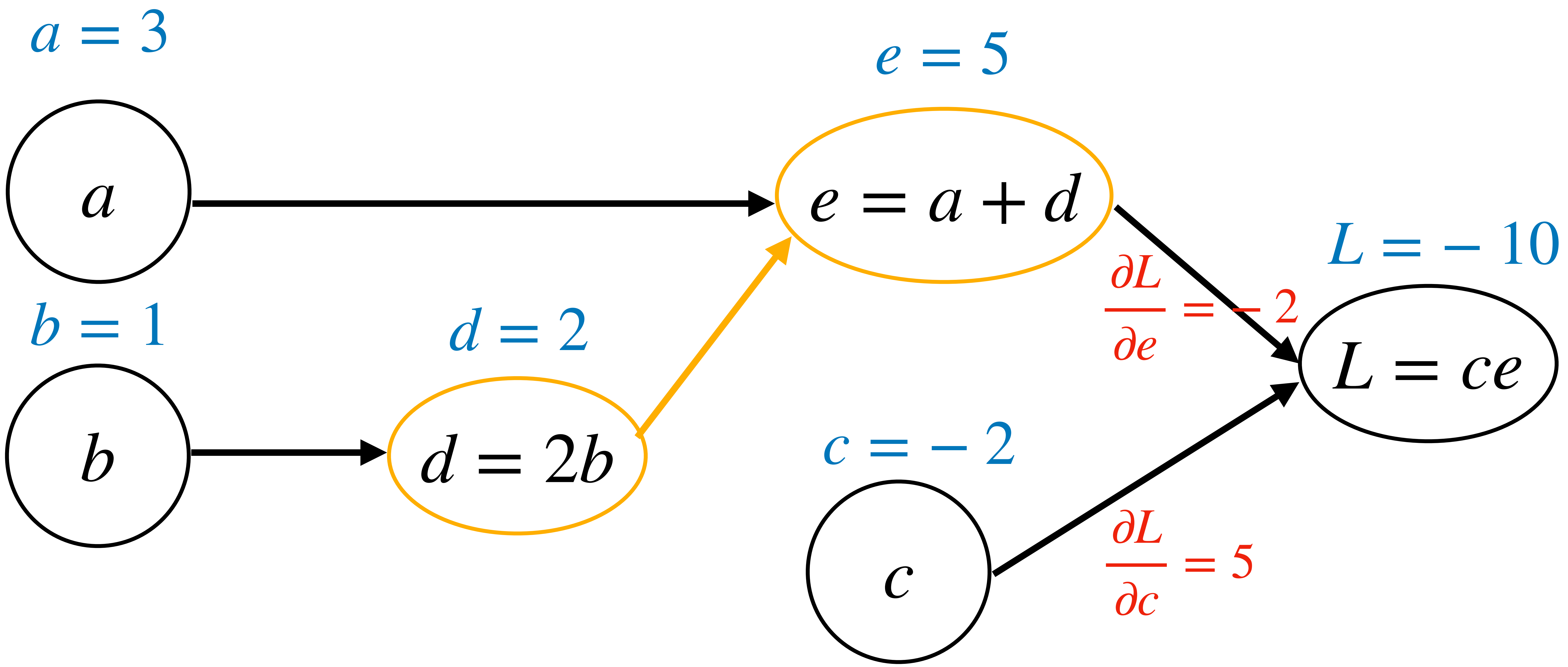

$e = a + d$

Backpropagation

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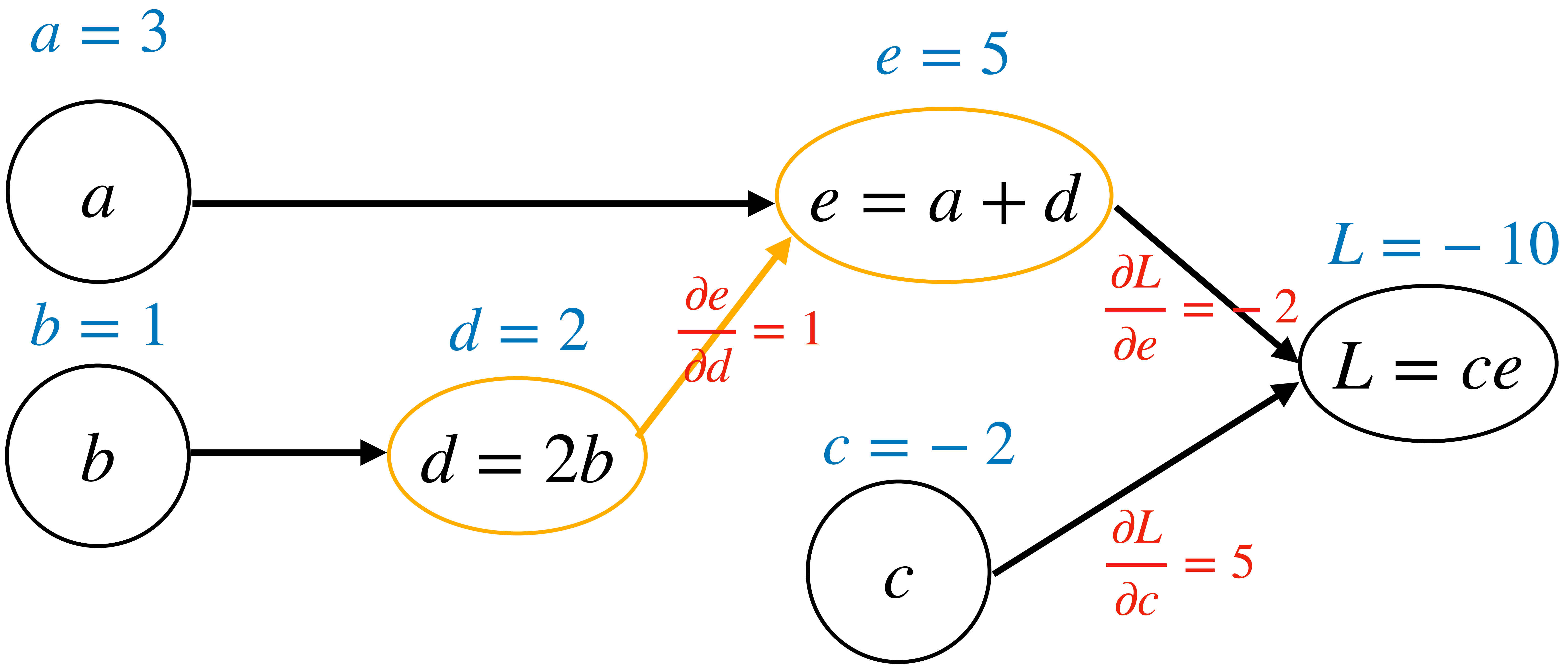

$$e = a + d$$

$$\Rightarrow \frac{\partial e}{\partial d} = 1$$

Backpropagation

Responsibility attribution through gradient

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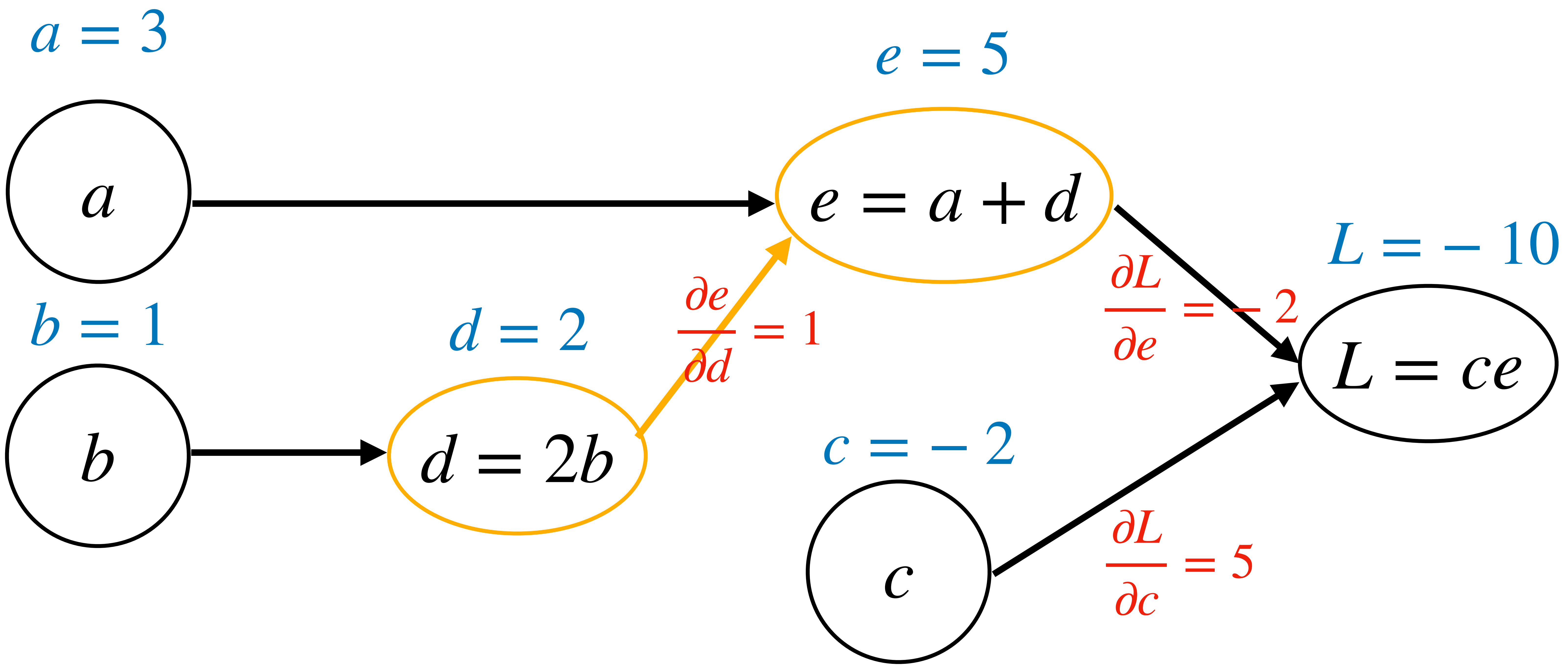

The Chain Rule

$$f(x) = u(v(w(x)))$$

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$e = a + d$$

$$\Rightarrow \frac{\partial e}{\partial d} = 1$$

Backpropagation

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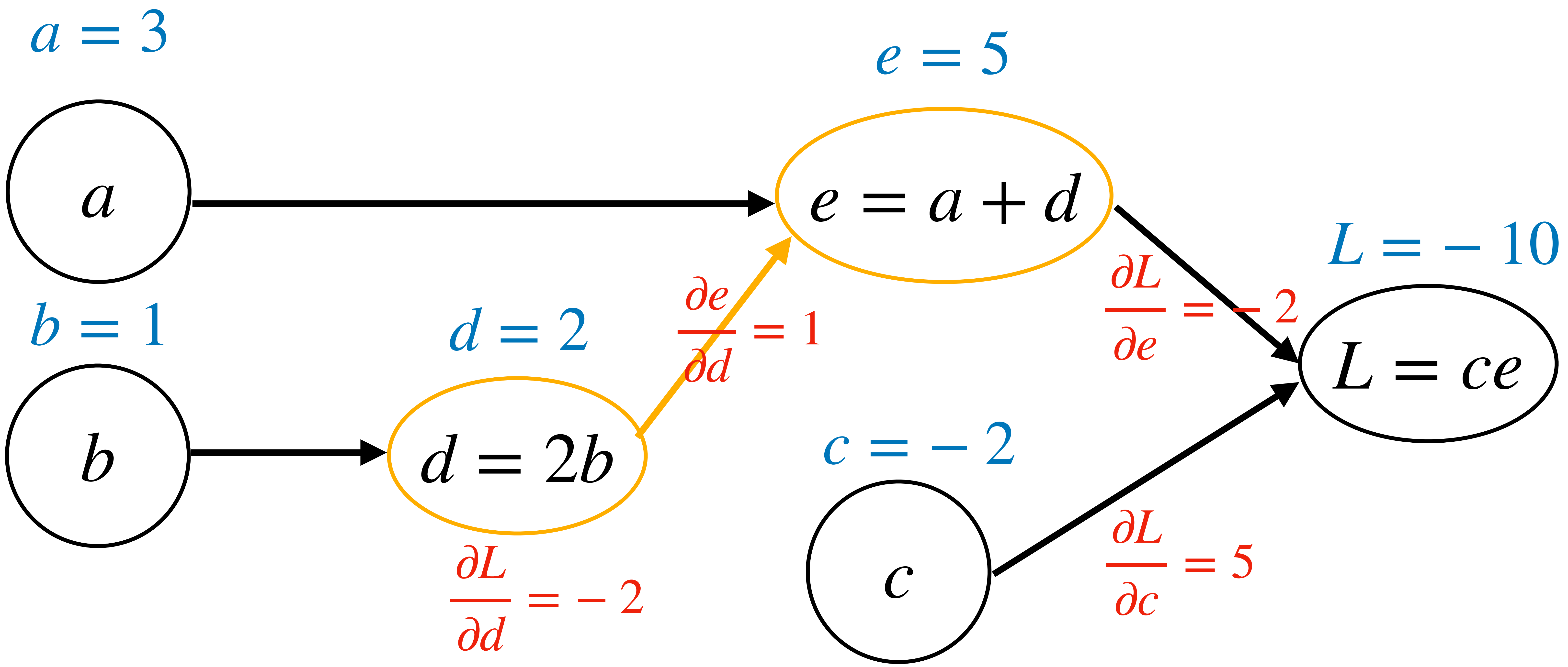

$$e = a + d$$

$$\Rightarrow \frac{\partial e}{\partial d} = 1$$

$$\Rightarrow \frac{\partial L}{\partial d} = \frac{\partial L}{\partial e}\frac{\partial e}{\partial d}$$

$$= 1 \times -2$$

$$= -2$$

Backpropagation

Responsibility attribution through gradient

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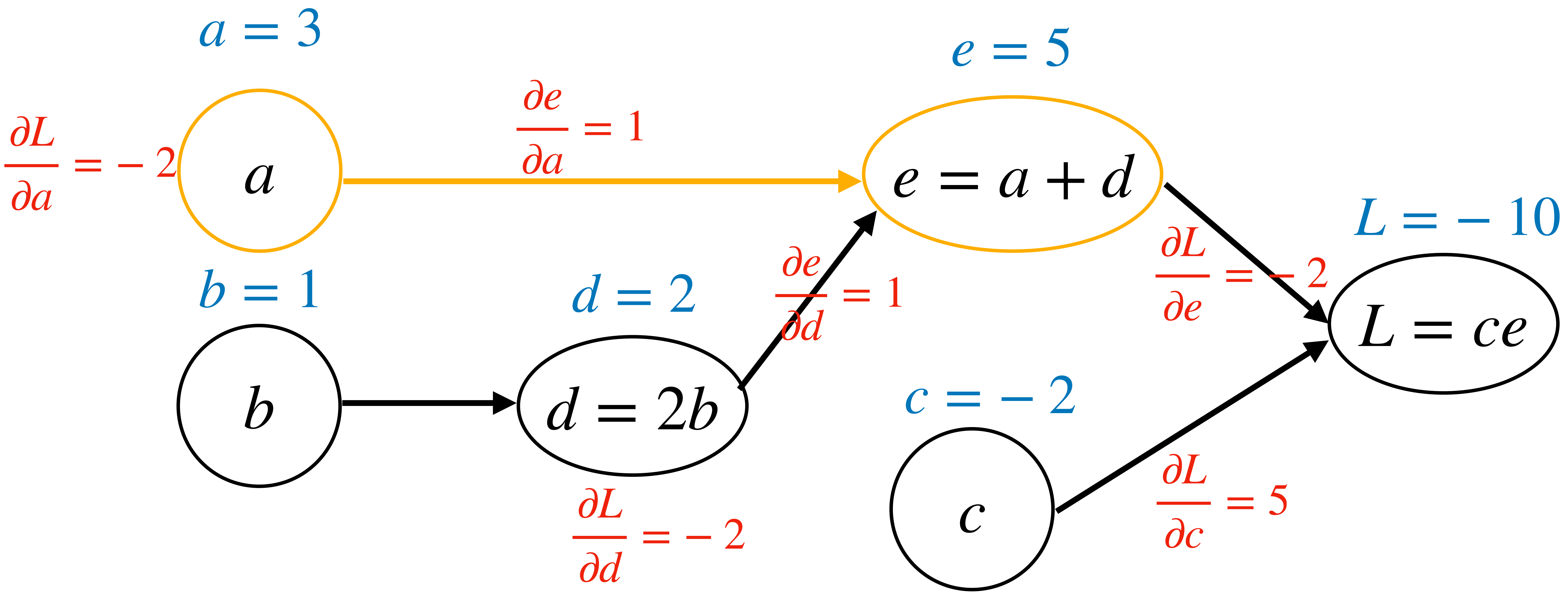

$$e = a + d$$

$$\Rightarrow \frac{\partial e}{\partial a} = 1$$

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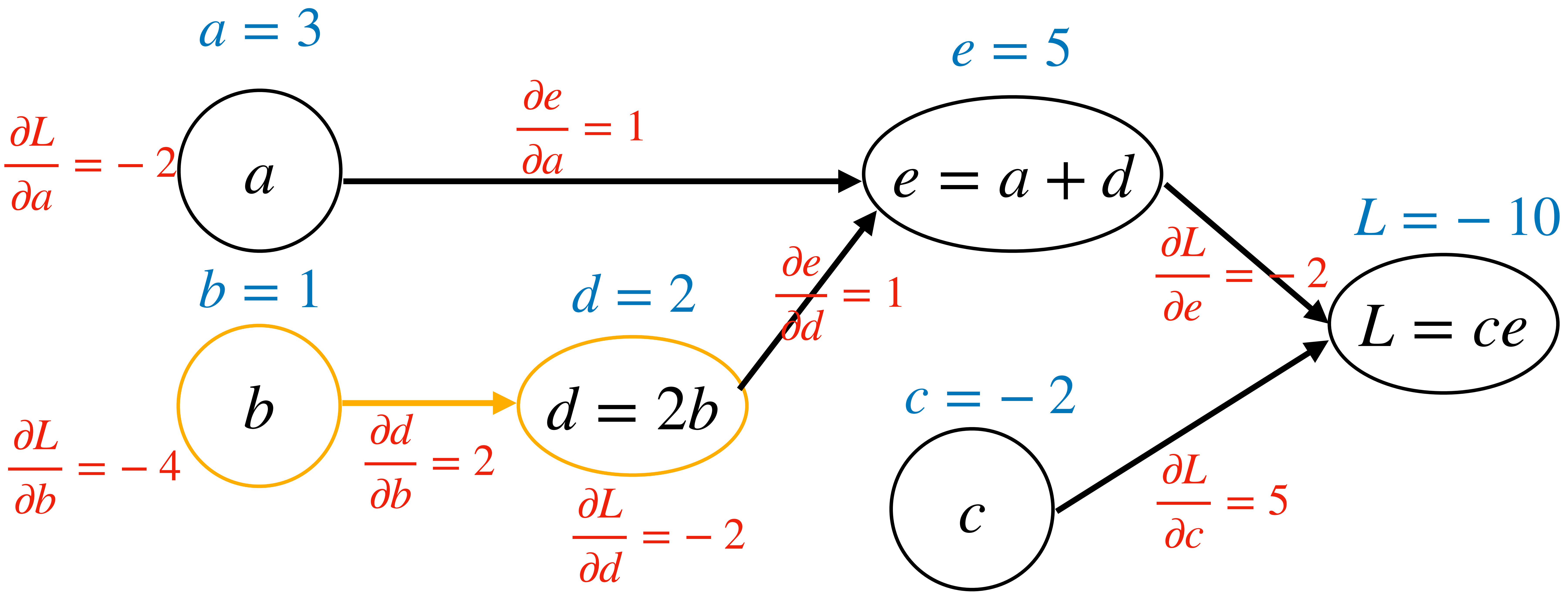

$$d = 2b$$

$$\Rightarrow \frac{\partial d}{\partial b} = 2$$

$$\Rightarrow \frac{\partial L}{\partial b} = \frac{\partial L}{\partial e}\frac{\partial e}{\partial d}\frac{\partial d}{\partial b}$$

$$= -2 \times 1 \times 2$$

$$= -4$$

Backpropagation

Responsibility attribution through gradient

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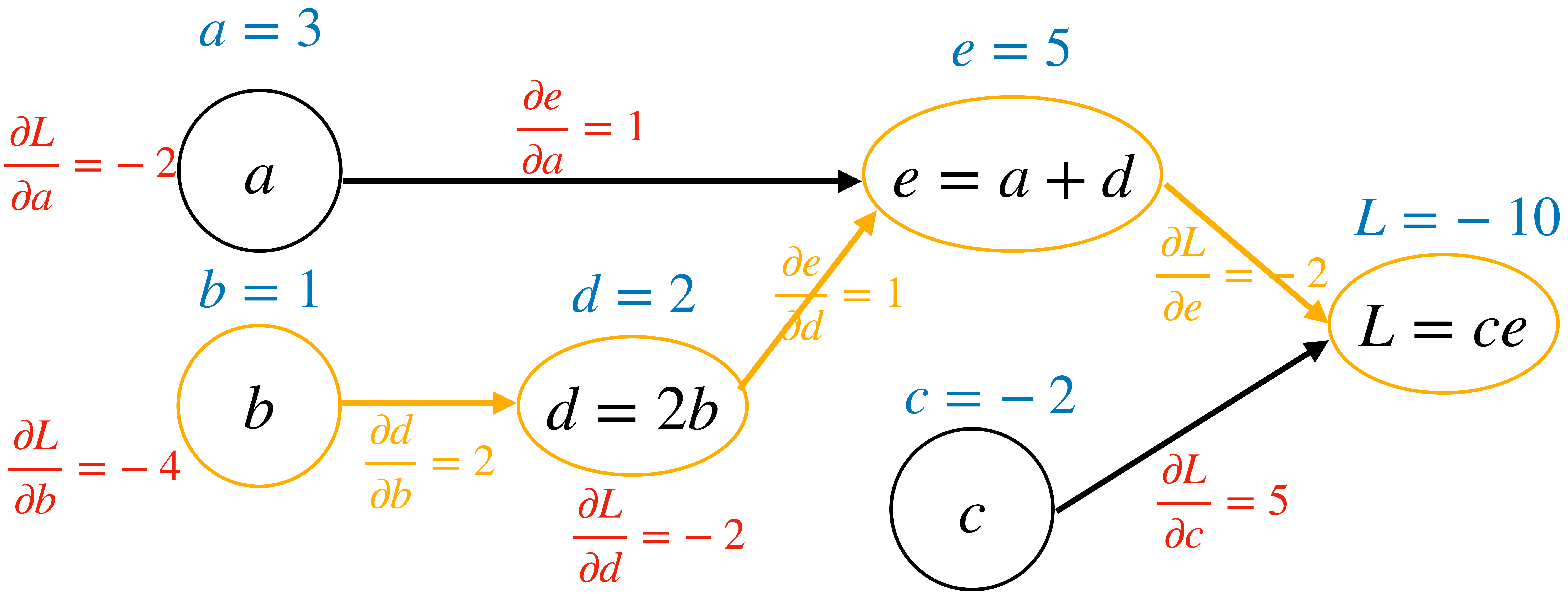
- $\underline{L(a,b,c) = c(a+2b)}$. Now let $a = 3, b = 1, c = -2$;
  - **Derivative on an edge**: *local* dependency;
  - **Derivative on node**: *long* dependency all the way from the final node (L)



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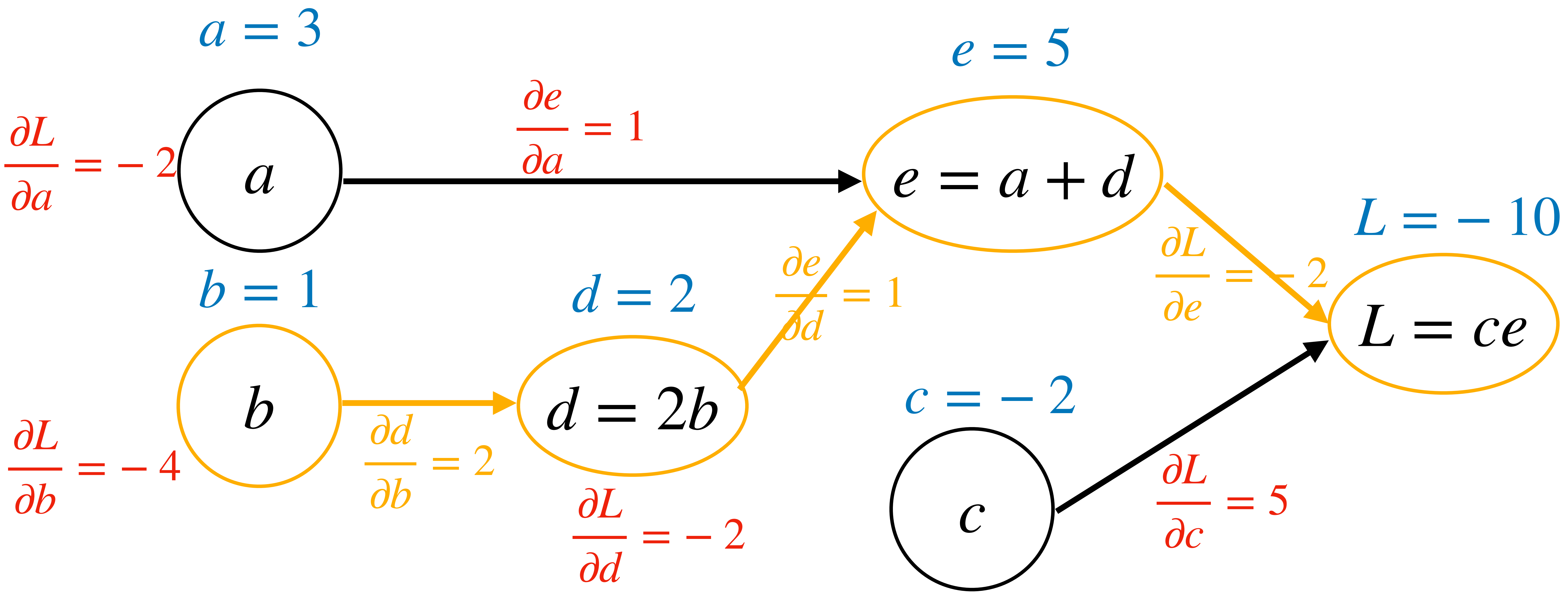

$d = 2b$

Edge: local

$$\Rightarrow \frac{\partial d}{\partial b} = 2$$

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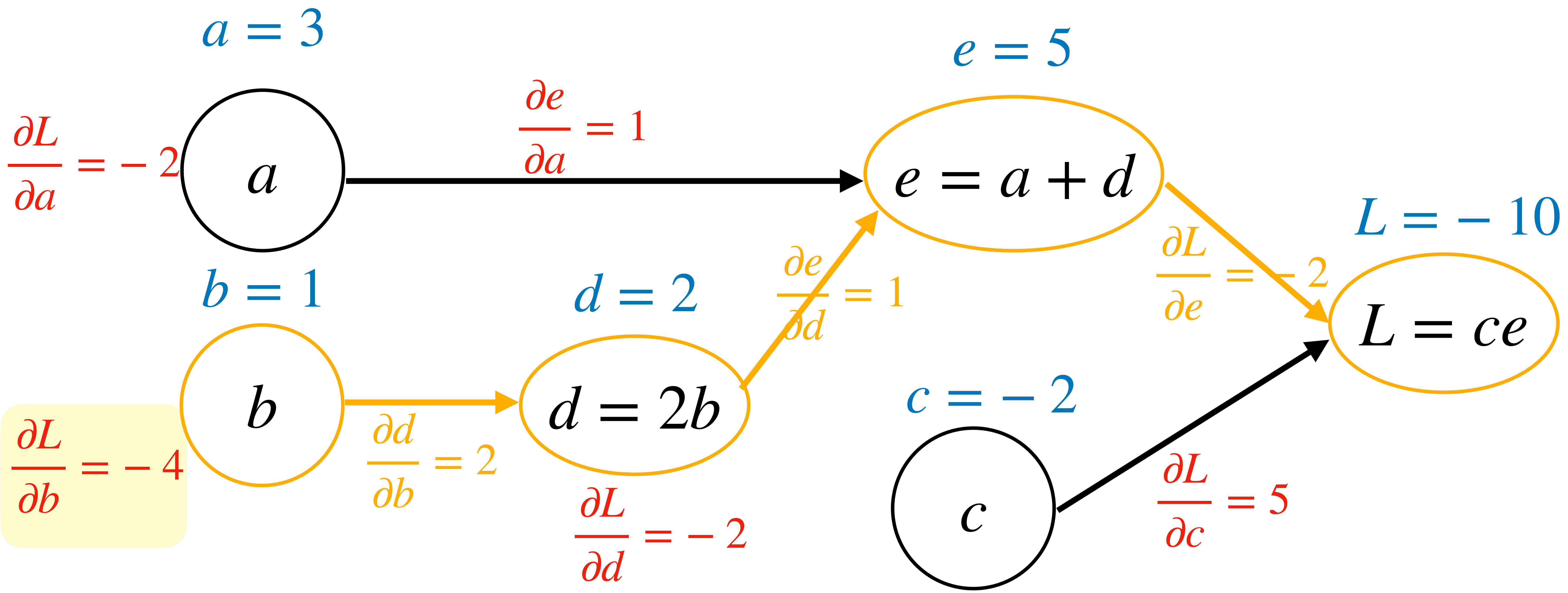


$d = 2b$

$\Rightarrow \frac{\partial d}{\partial b} = 2$ **Edge: local**

$$\Rightarrow \frac{\partial L}{\partial b} = \frac{\partial L}{\partial e}\frac{\partial e}{\partial d}\frac{\partial d}{\partial b}$$

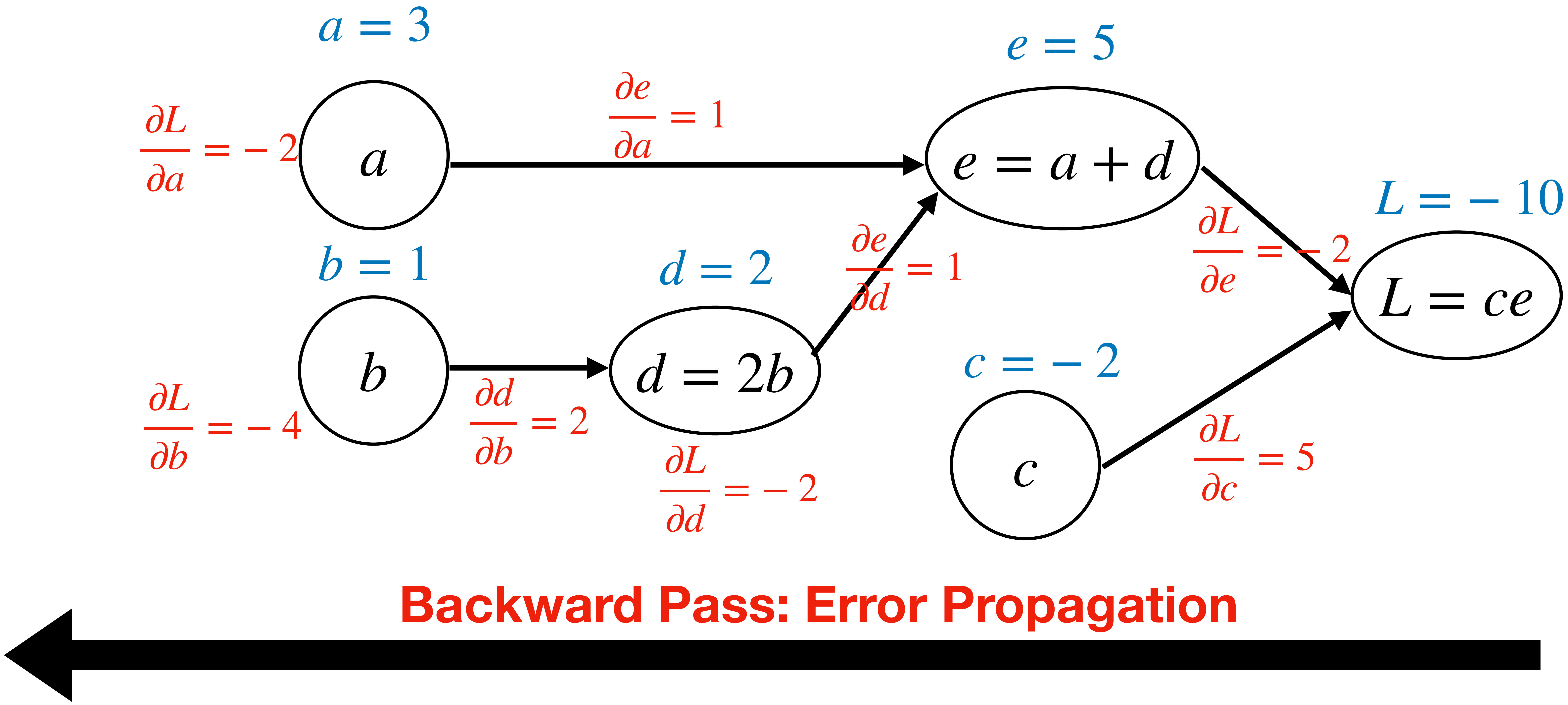
Node: long $= -2 \times 1 \times 2$

$= -4$

Backpropagation → Gradient Descent

Learning/Updating the weights to minimize loss

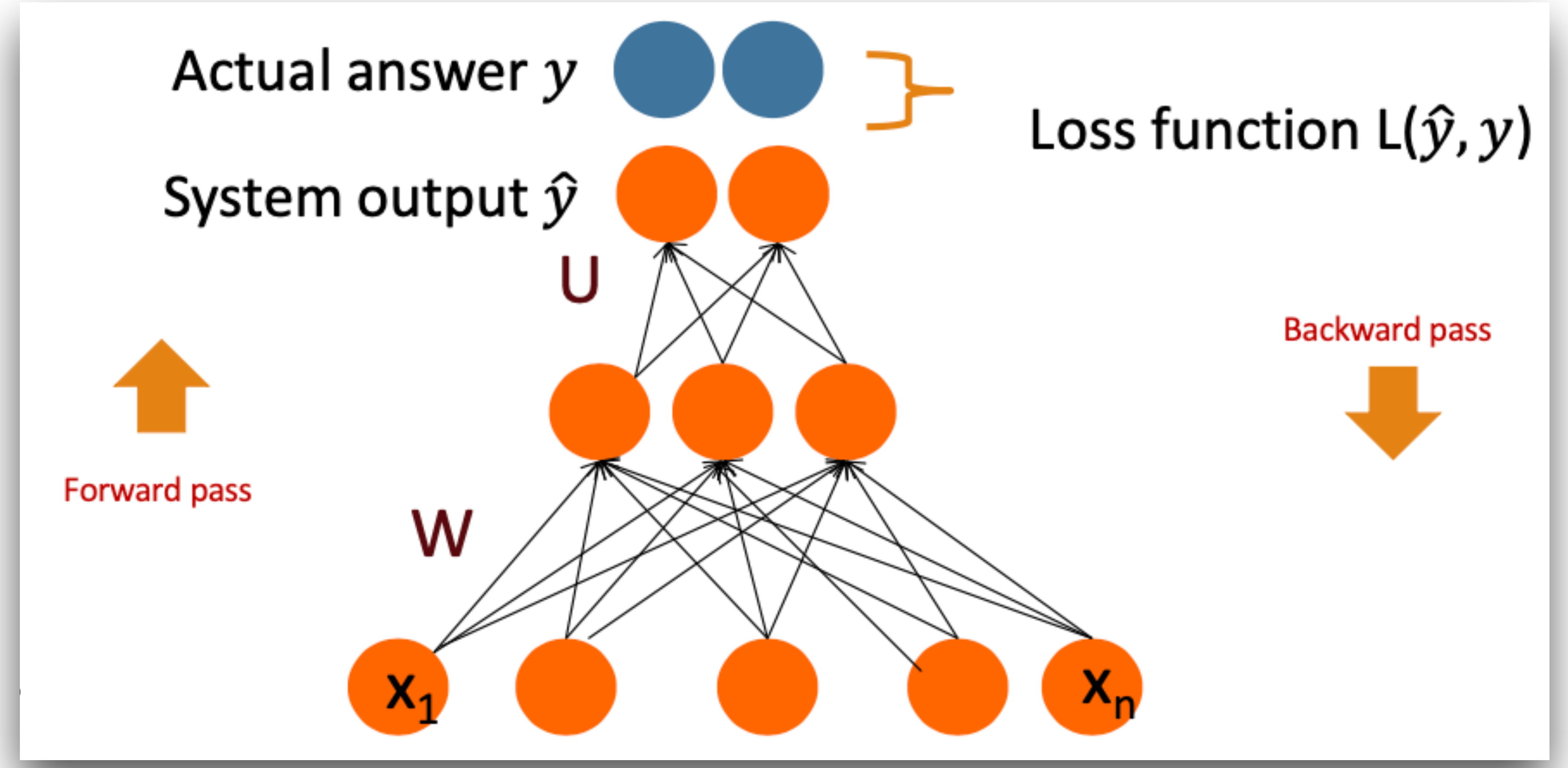
Update each weight: $w' = w - \eta \frac{d}{dw} L(f(x;w), y)$, where $L(f(x;w), y) = L(\hat{y} - y)$



Gradient Descent in Neural Networks

For every training tuple (x, y)

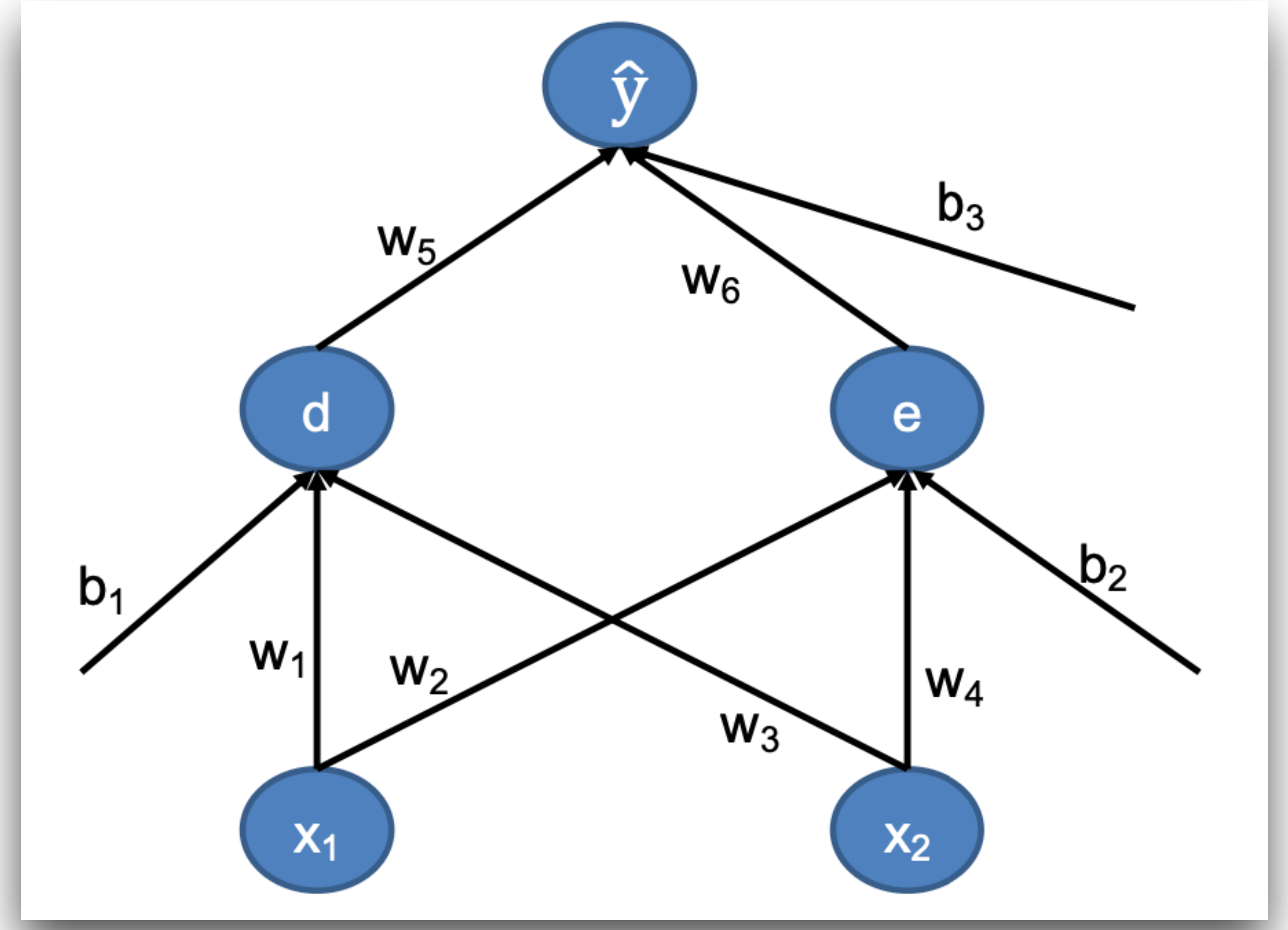
- Run *forward* computation to find our estimate $\hat{y}$
- Run *backward* computation to update weights:
  - For every output node
    - Compute loss L between true y and the estimated $\hat{y}$
    - For every weight w from hidden layer to the output layer
      - Update the weight
  - For every hidden node
    - Assess how much blame it deserves for the current answer
    - For every weight w from input layer to the hidden layer
      - Update the weight



The computation quickly blows up

Verify this when you are free!

For a simple two-layer neural network:



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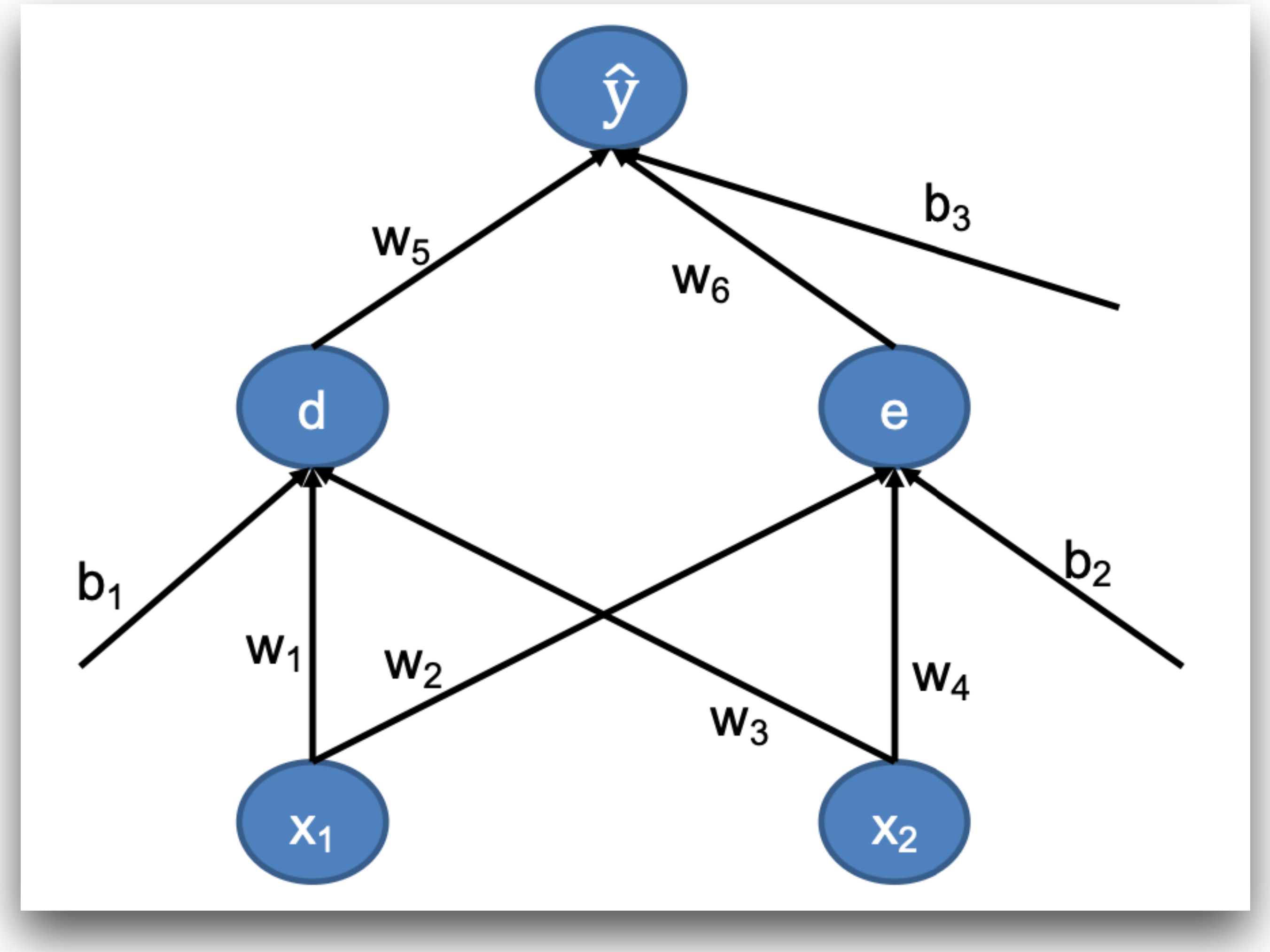
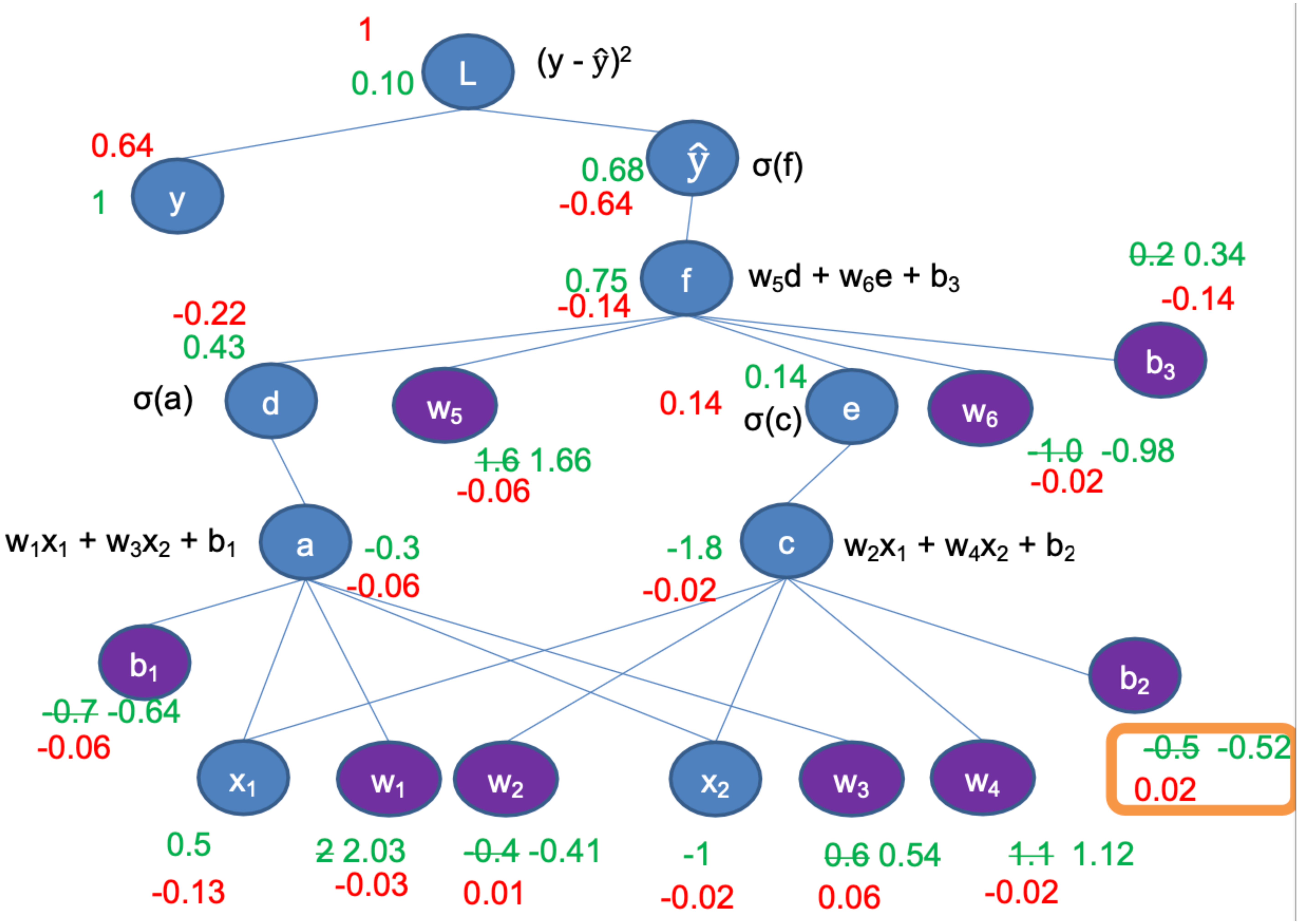


One round of gradient descent:



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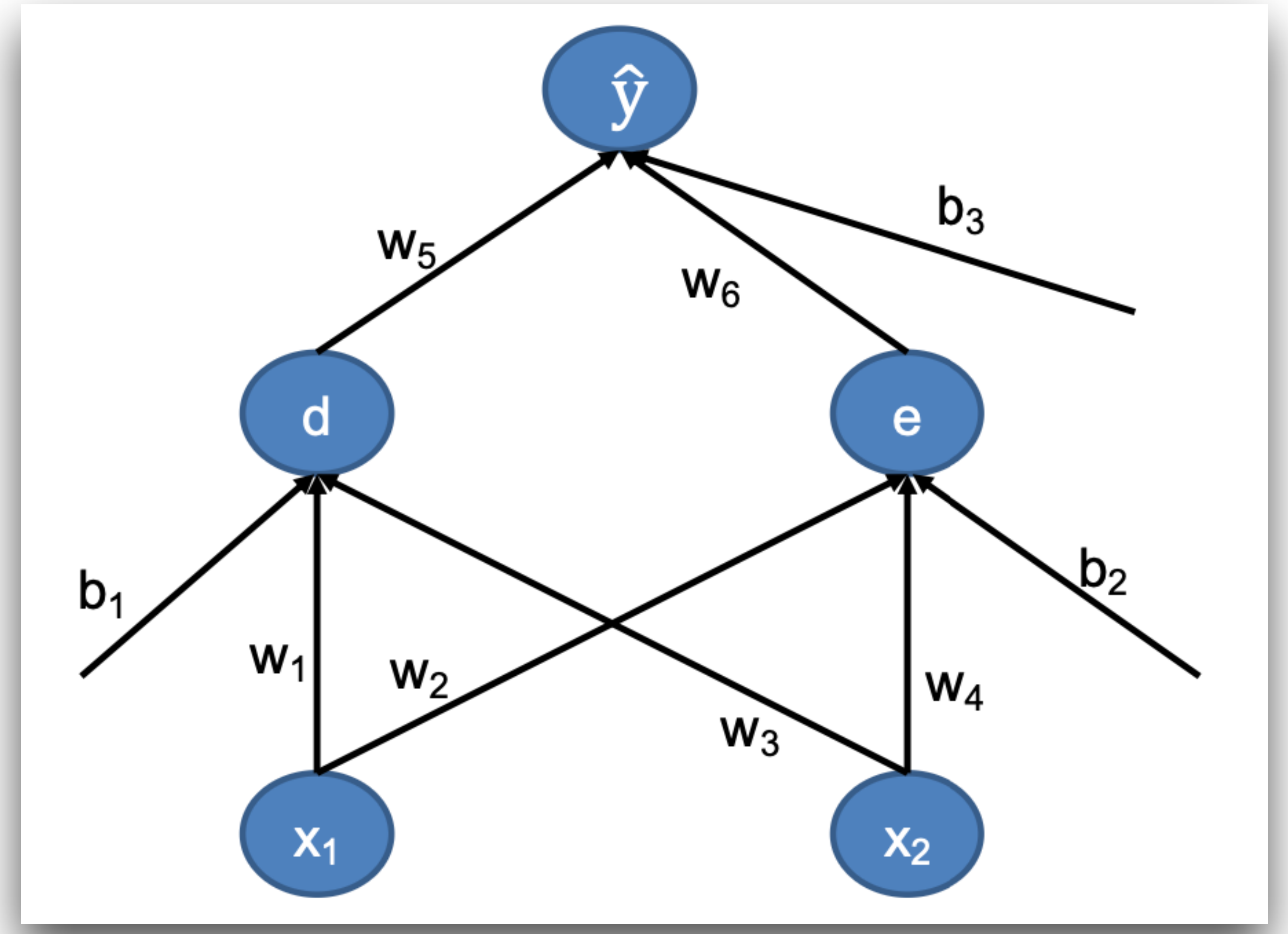


One round of gradient descent:

Example: Binary logistic regression

- $L(\hat{y}, y) = \ln(1 - \frac{1}{1+ e^{-(w_1x_1+w_2x_2+b)}})$ — Repeat from previous slide
- $L(\hat{y}, y) = \ln(\frac{1+ e^{-(w_1x_1+w_2x_2+b)}}{1+ e^{-(w_1x_1+w_2x_2+b)}} - \frac{1}{1+ e^{-(w_1x_1+w_2x_2+b)}})$ — Algebra
- $L(\hat{y}, y) = \ln(\frac{e^{-(w_1x_1+w_2x_2+b)}}{1+ e^{-(w_1x_1+w_2x_2+b)}})$ — Algebra
- $L(\hat{y}, y) = \ln\left(e^{-(w_1x_1+w_2x_2+b)}\right) - \ln(1 + e^{-(w_1x_1+w_2x_2+b)})$ — Algebra
- $L(\hat{y}, y) = -(w_1x_1 + w_2x_2 + b) - \ln(1 + e^{-(w_1x_1+w_2x_2+b)})$ — Algebra

- $\frac{\partial}{\partial w_1} L(\hat{y}, y) = -x_1 - \frac{1}{1+ e^{-(w_1x_1+w_2x_2+b)}} e^{-(w_1x_1+w_2x_2+b)}(-x_1)$ — Take partial derivative
- $\frac{\partial}{\partial w_1} L(\hat{y}, y) = x_1(-1 + \frac{e^{-(w_1x_1+w_2x_2+b)}}{1+ e^{-(w_1x_1+w_2x_2+b)}})$ — Algebra
- $\frac{\partial}{\partial w_1} L(\hat{y}, y) = x_1(\frac{-1}{1+ e^{-(w_1x_1+w_2x_2+b)}})$ — Algebra

Notes on Loss Functions

Bonus on math

- If you wonder why perceptron learning uses $+$ while neural network uses $-$ to update their weights….!

Perceptron: $\mathbf{w} = \mathbf{w} + \eta(y - \hat{y})\mathbf{x}$; Neural Network: $w' = w - \eta \frac{d}{dw} L(f(x; w), y)$

- **Short answer: they are actually the same thing!** Here is a derivation:

✳ For loss function $L = \frac{1}{2}(y - \hat{y})^2$ or $L = -\left[y \log(\hat{y}) + (1 - y)\log(1 - \hat{y}) \right]$:

✳ The gradient with respect to weight w_i turns out to be: $\frac{\partial L}{\partial w_i} = -(y - \hat{y})\, x_i$

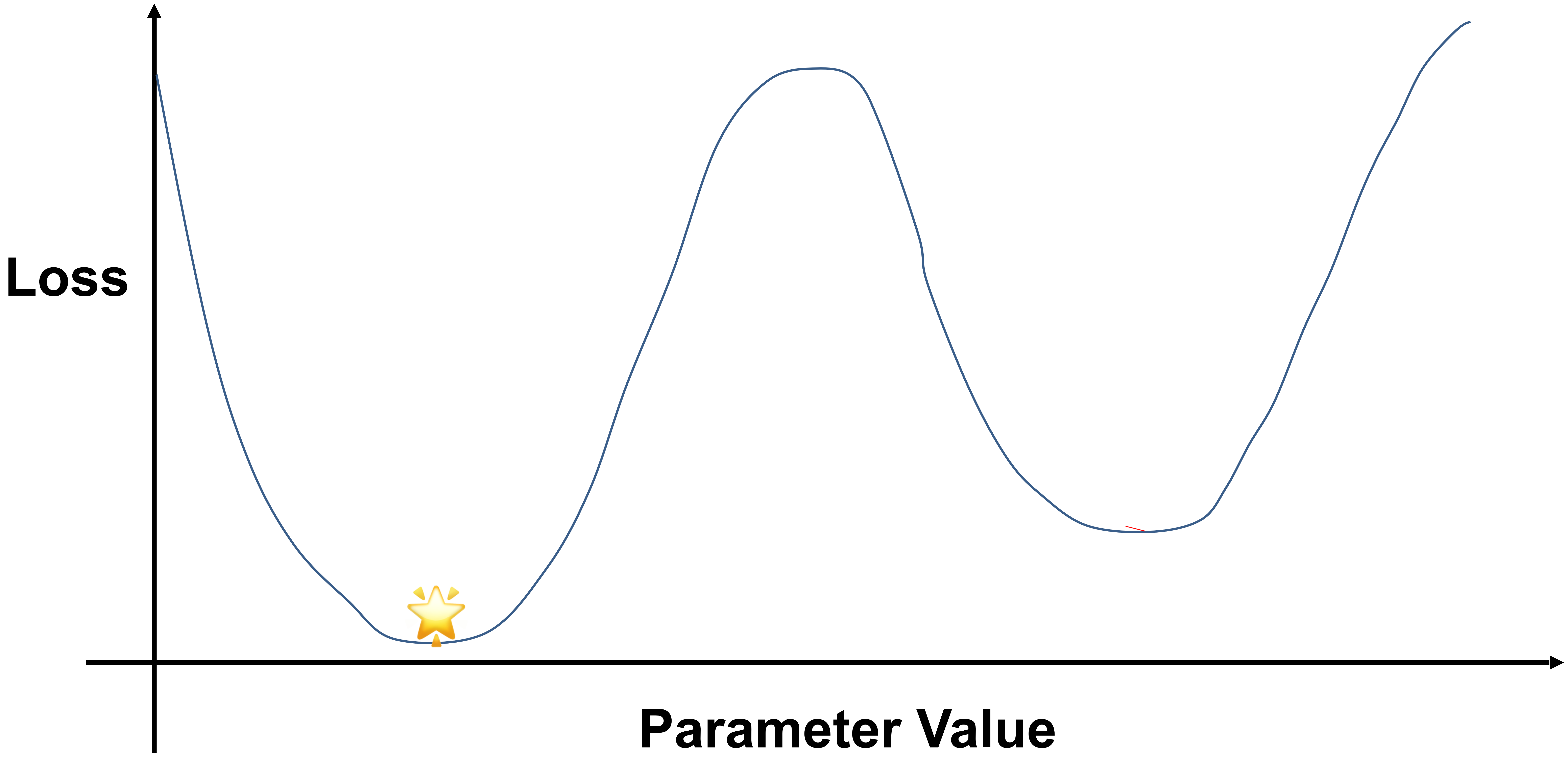
✳ If we plug this into the gradient descent formula:

$$\mathbf{w} := \mathbf{w} - \eta \frac{\partial L}{\partial \mathbf{w}} = \mathbf{w} - \eta \left(-(y - \hat{y})\, \mathbf{x} \right) = \mathbf{w} + \eta(y - \hat{y})\, \mathbf{x}$$

✳ The two minus signs cancels, giving the same direction as perception update.

Gradient Descent: Intuitions in 1D Space

Learning to minimize loss (as an optimization problem)



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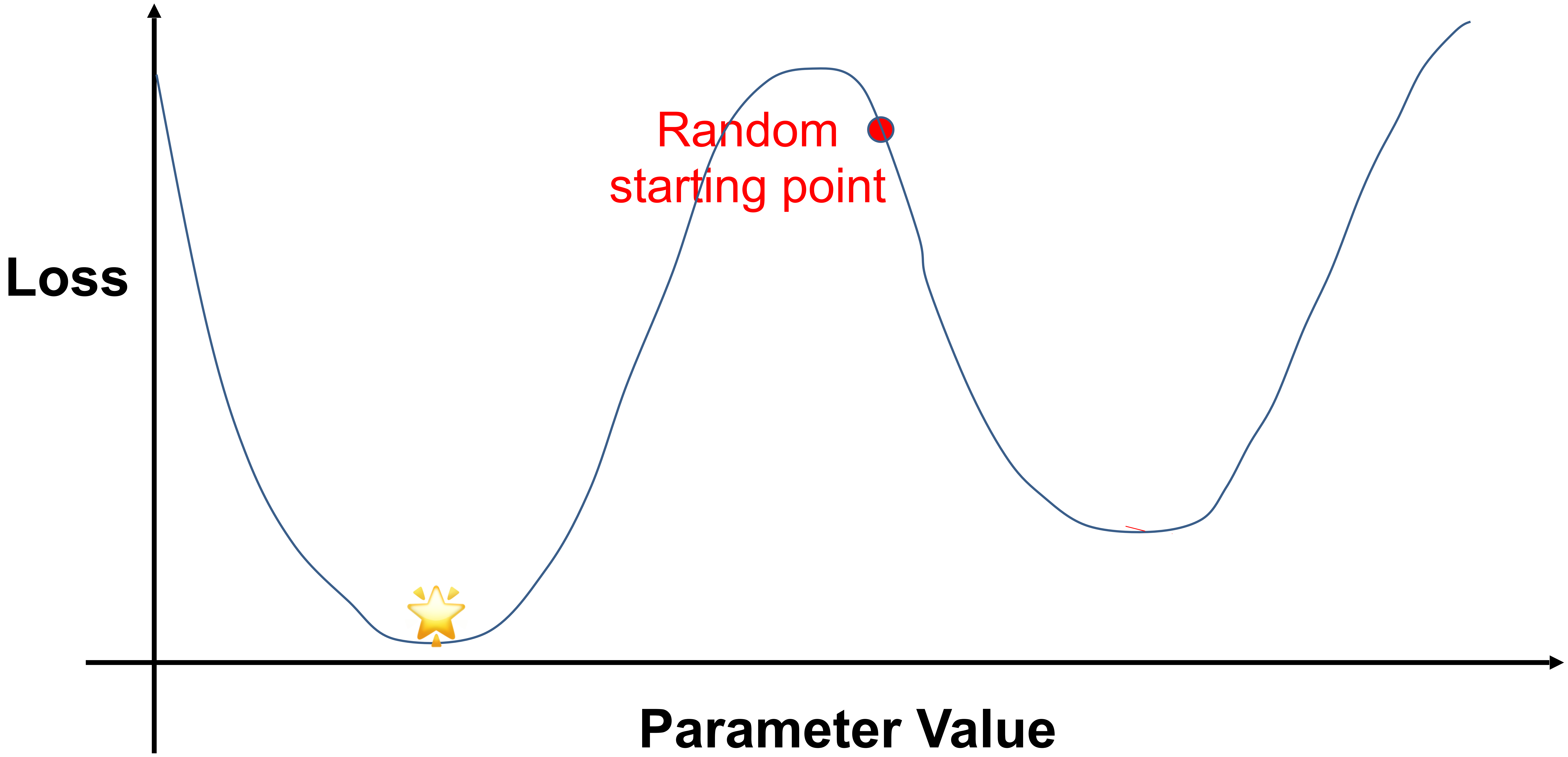

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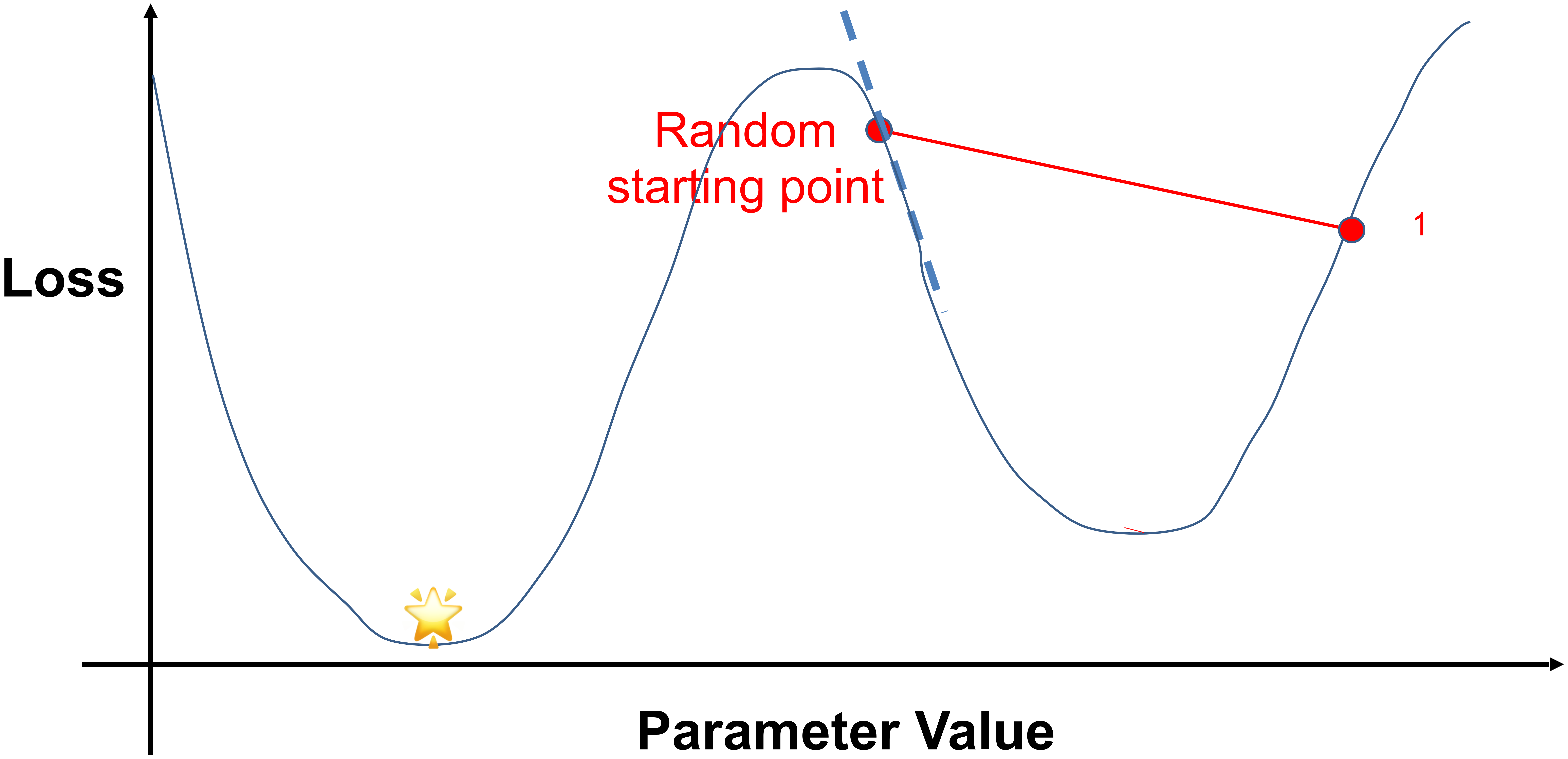

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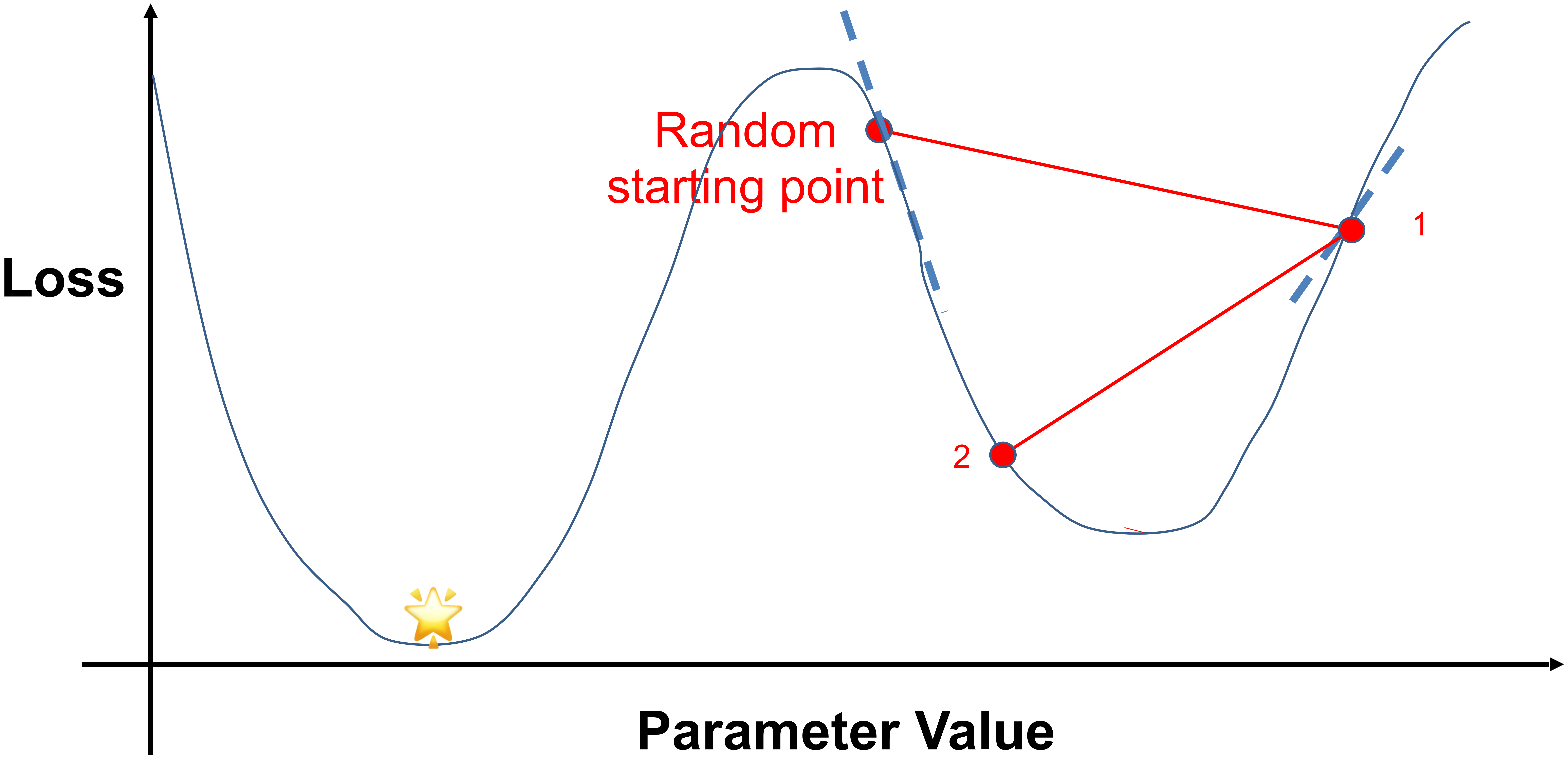

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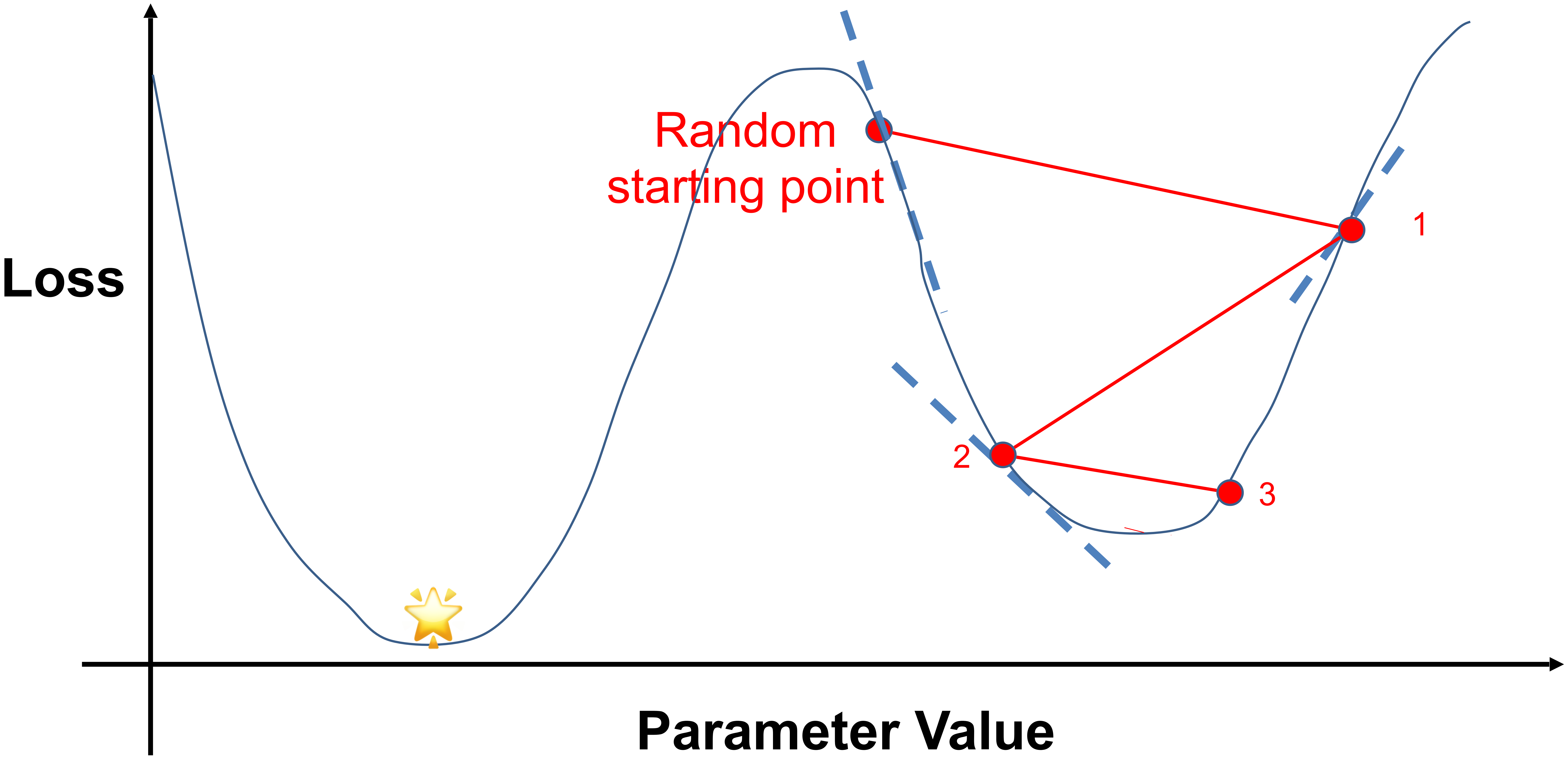
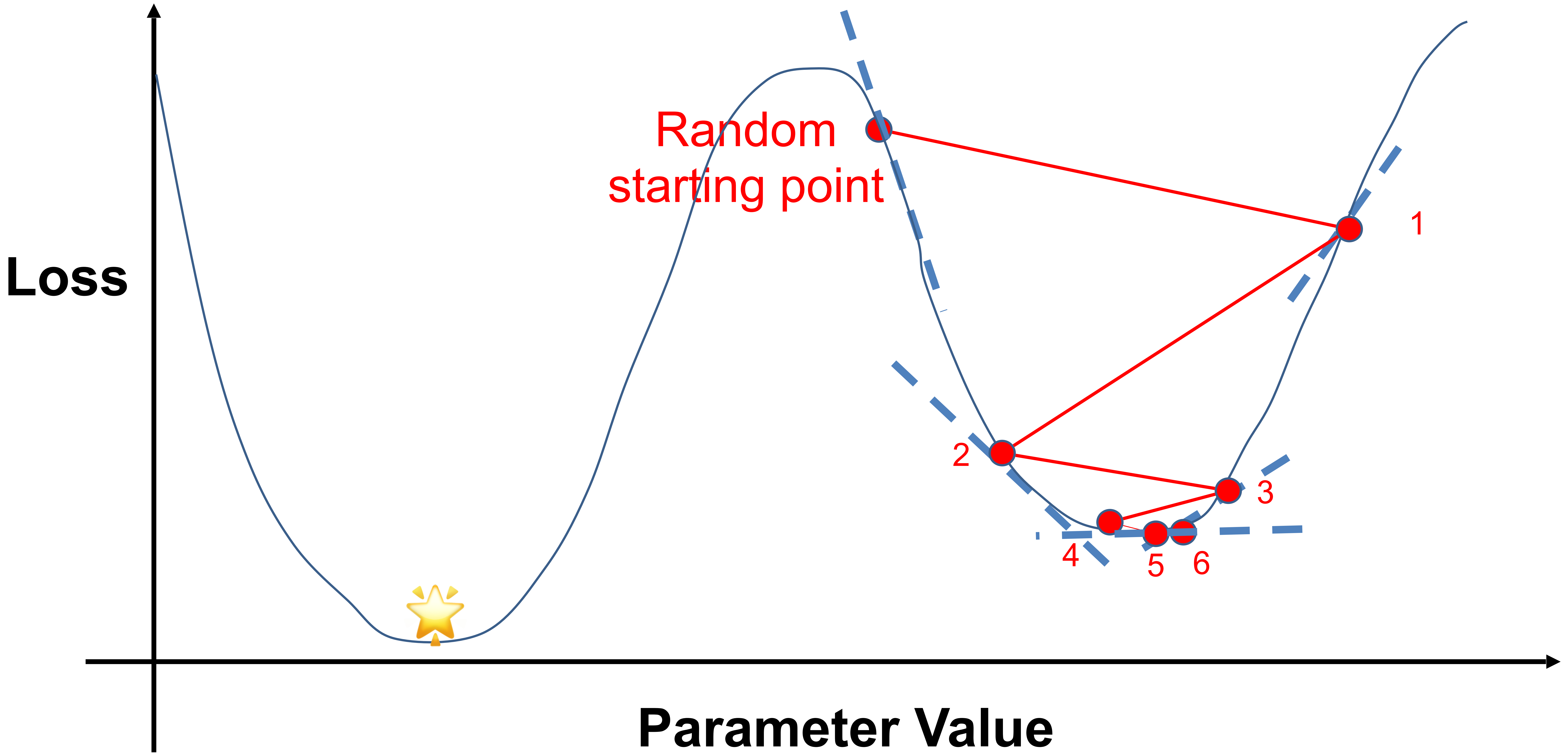


Gradient Descent: Intuitions in 1D Space

Learning to minimize loss (as an optimization problem)



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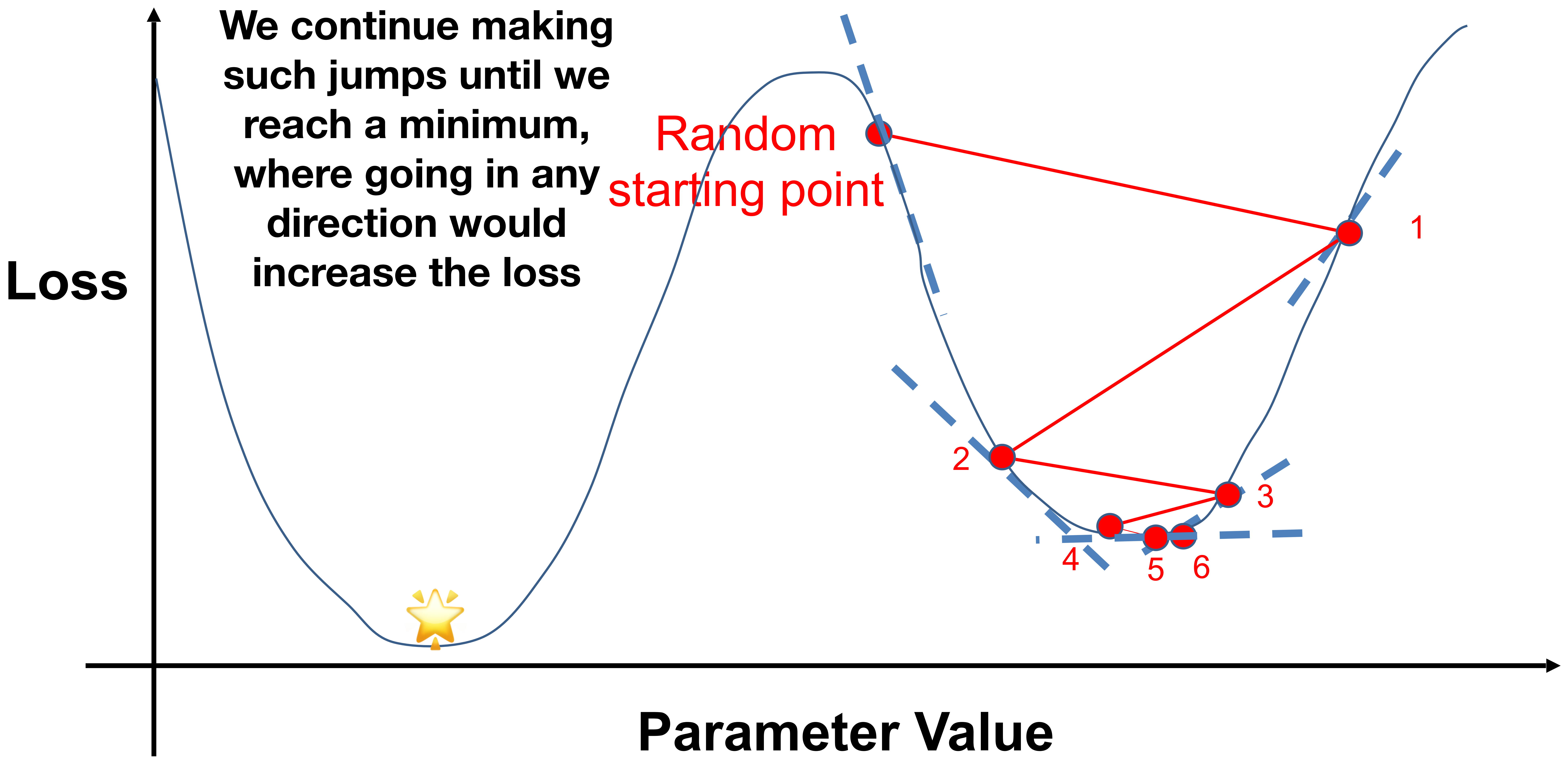
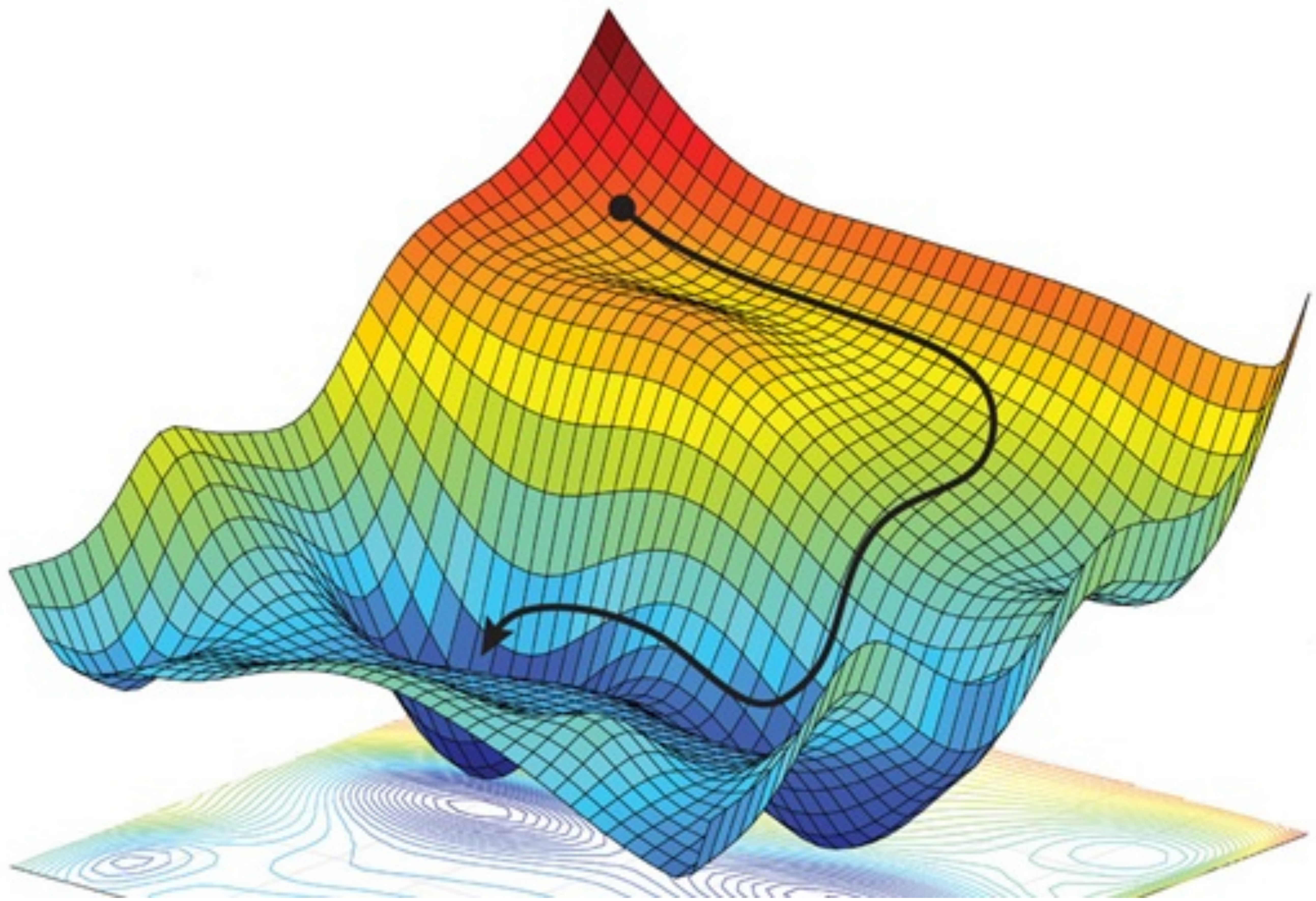


Gradient Descent: Intuitions in High Dimensions

Learning to minimize loss (as an optimization problem)



Interium Summary 4

Backpropagation and Gradient Descent

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- **Backpropagation** (a method for efficiently computing the gradients) tells us *what the gradients are* for each weight.
  - Computing dependencies across layers through chain rule in computation graphs;

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Backpropagation and Gradient Descent

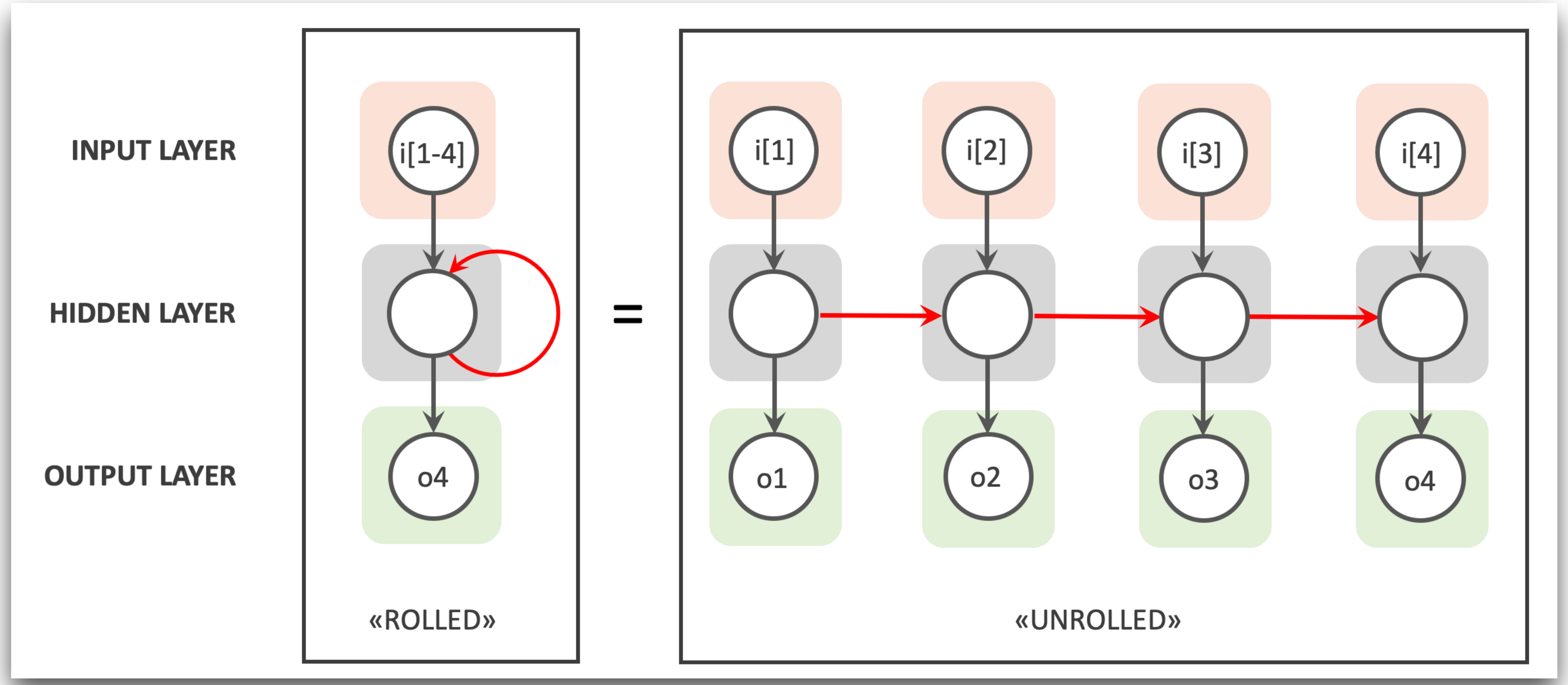
- **Backpropagation** (a method for efficiently computing the gradients) tells us *what the gradients are* for each weight.
  - Computing dependencies across layers through chain rule in computation graphs;
- **Gradient descent** (an optimization algorithm) tells us *how to update* the weight.
  - Iteratively optimizing to minimize the loss for each training example.
  - A forward pass (generating prediction) followed by a backward pass (computing the gradient and updating the weights).

Development of Neural Network Architectures

1990s - present

Recurrent Neural Network (RNN)

Introducing the notion of *Time*



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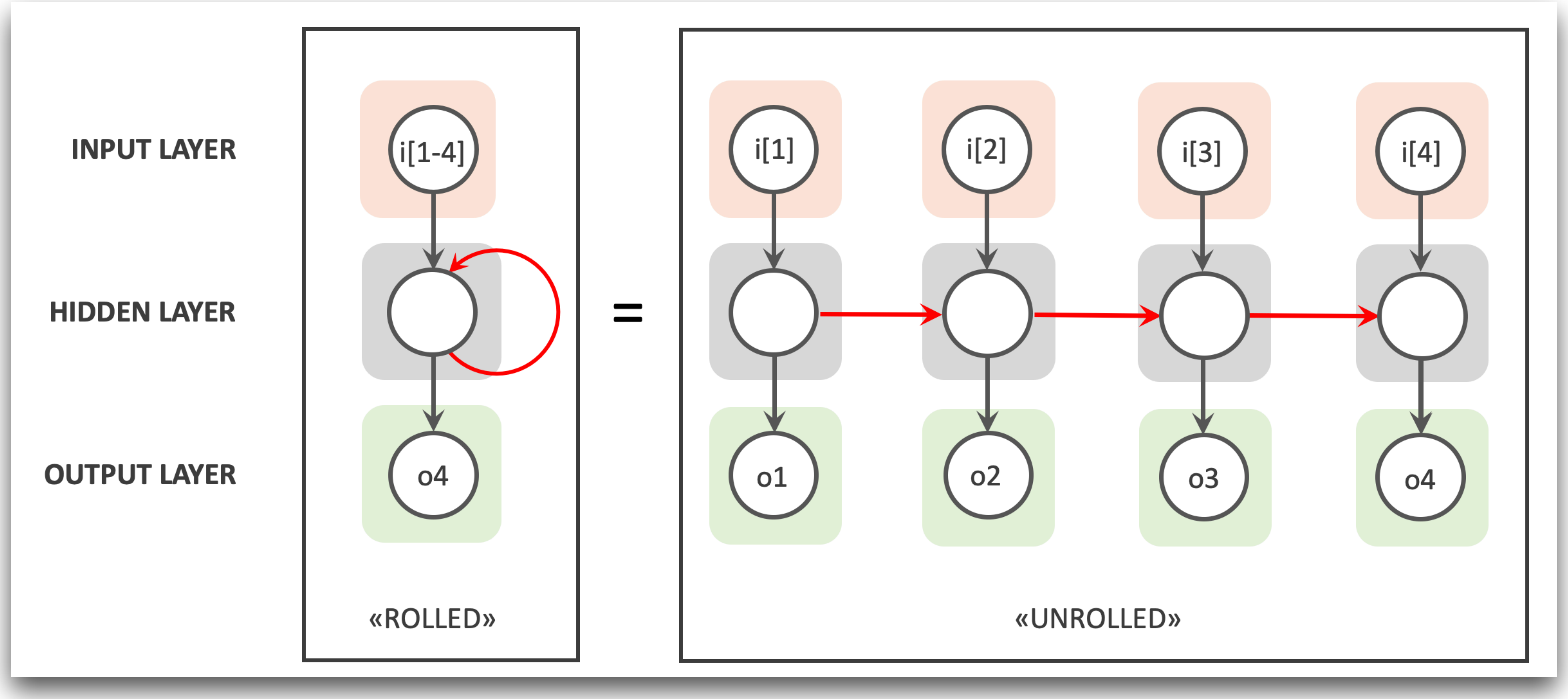

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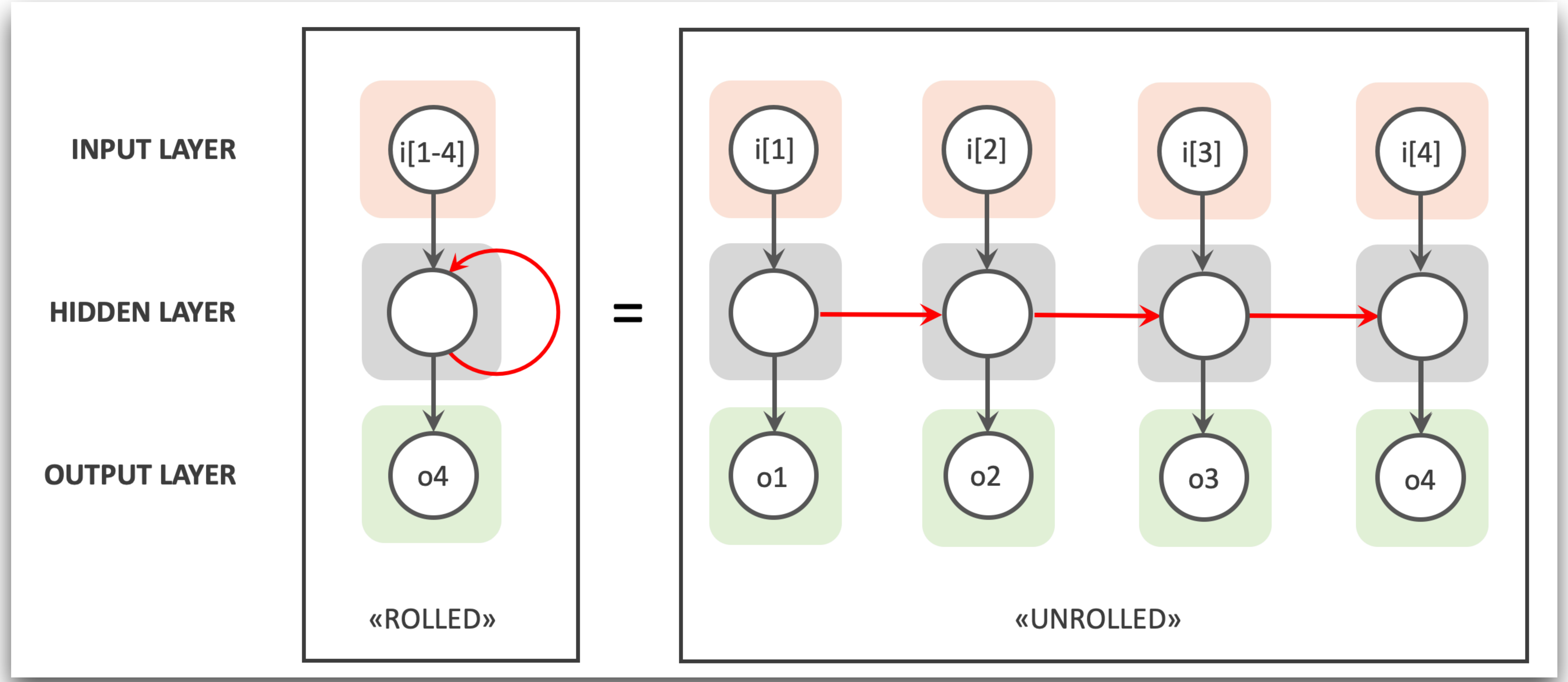
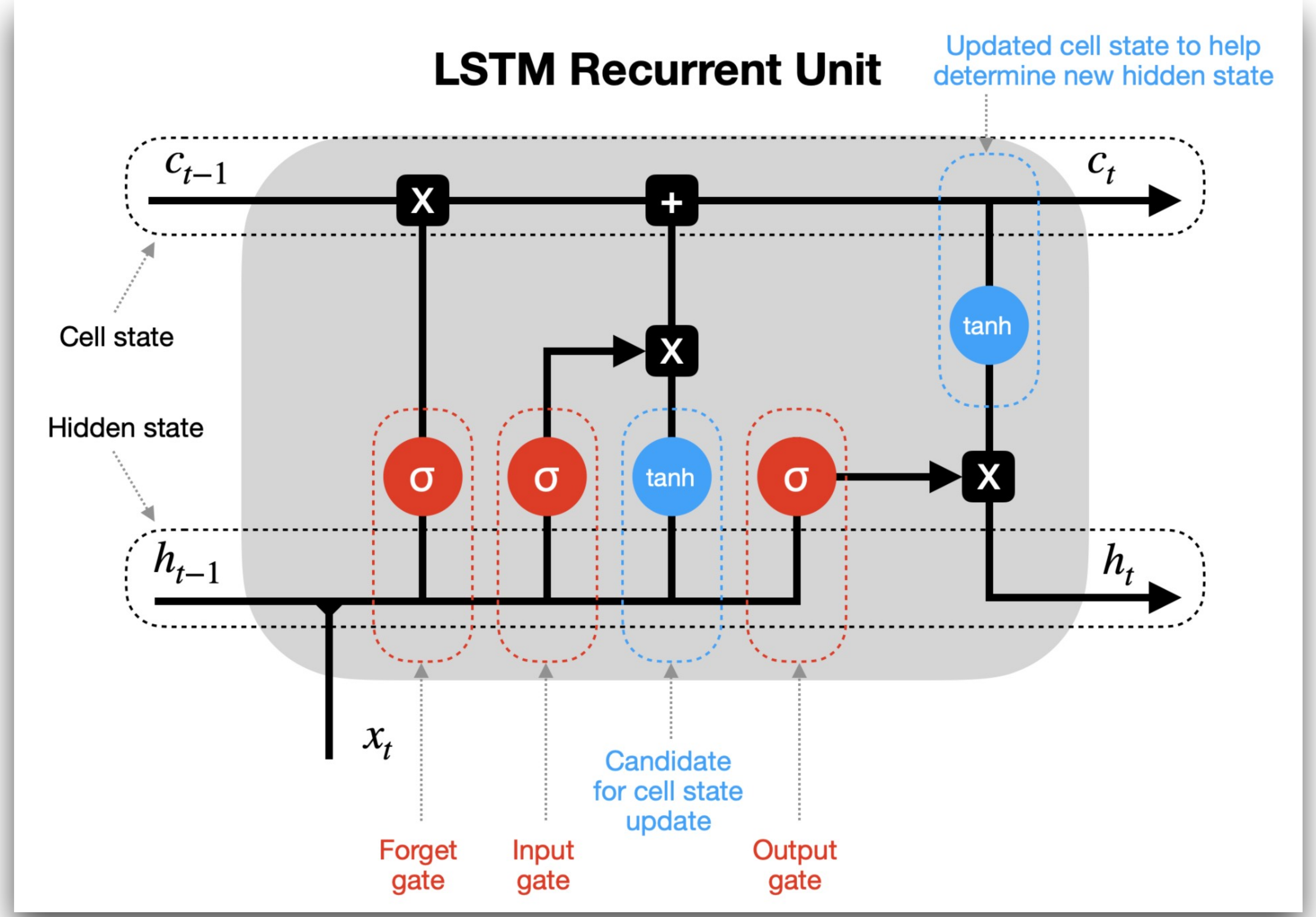


- **Motivation:** Standard feed-forward networks couldn't handle **sequential or time-dependent data**, since they treat all inputs as independent.
- **Usefulness:** RNNs introduce ***recurrent connections* that let information persist across time steps**, enabling modeling of language, speech, and temporal patterns.

Long Short-Term Memory (LSTM)

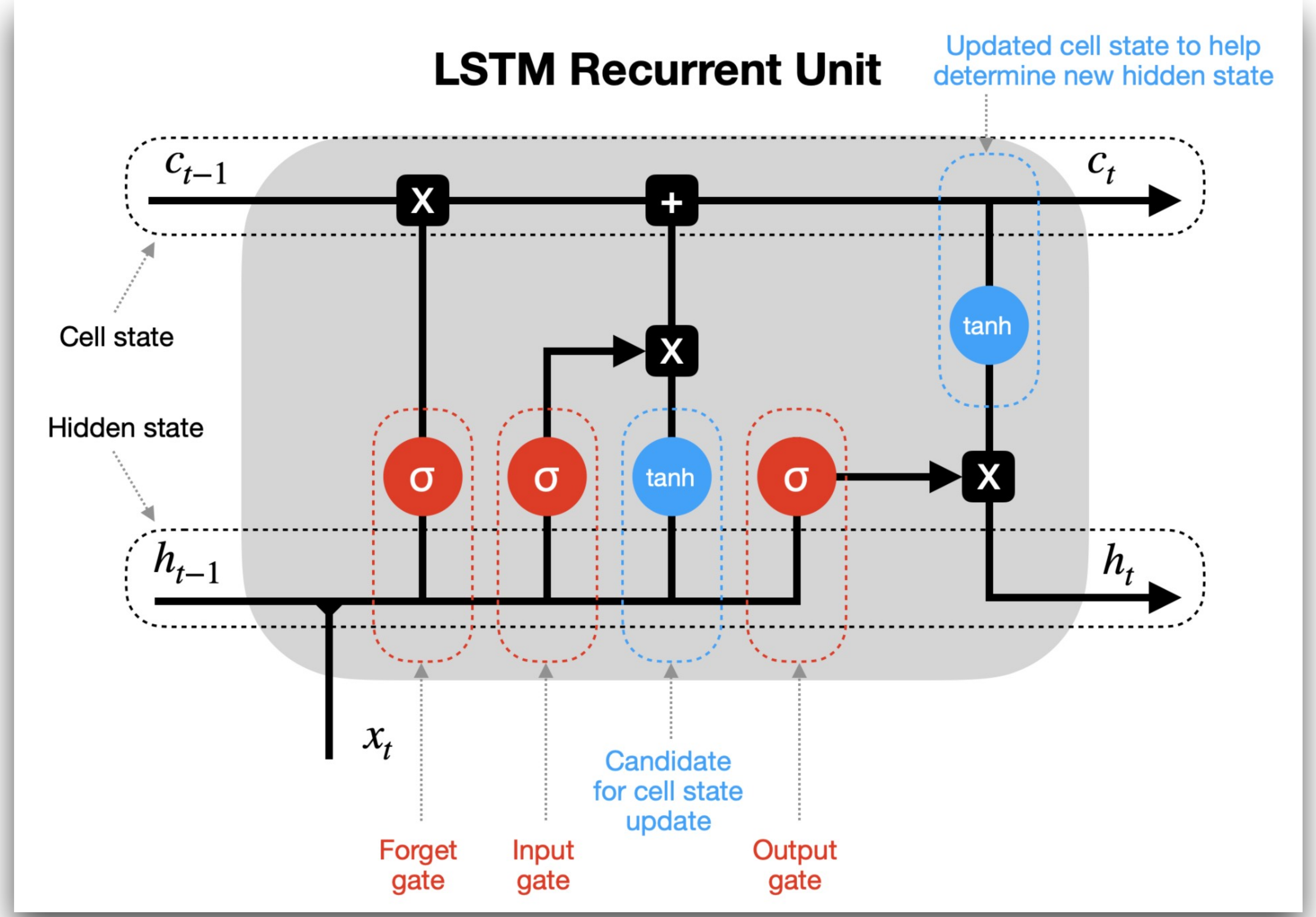
Handling long-distance dependencies through managing memory



Long Short-Term Memory (LSTM)

Handling long-distance dependencies through managing memory

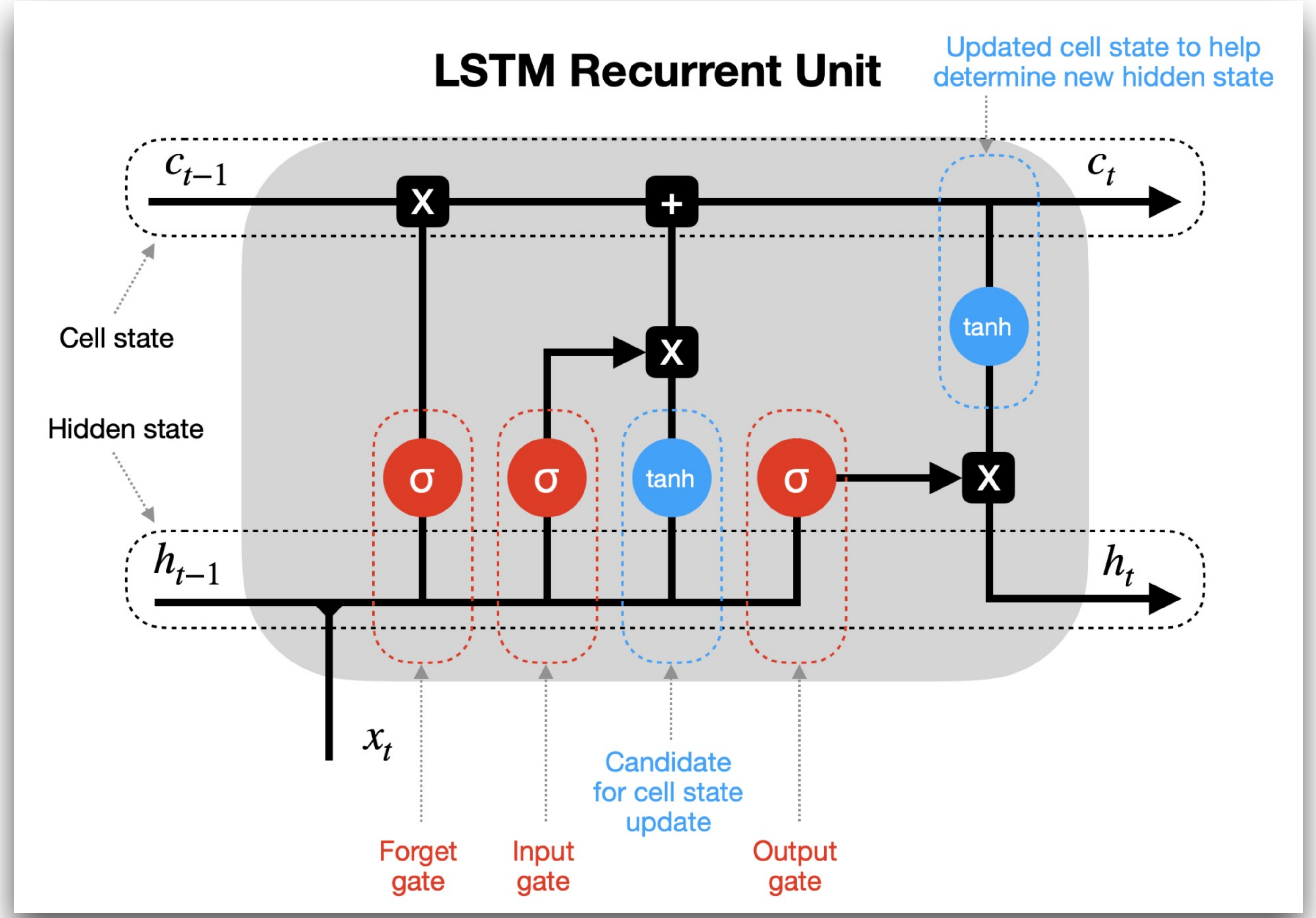
- **Motivation:** Vanilla RNNs struggled with *long-term dependencies* because of vanishing/exploding gradients during training.



Long Short-Term Memory (LSTM)

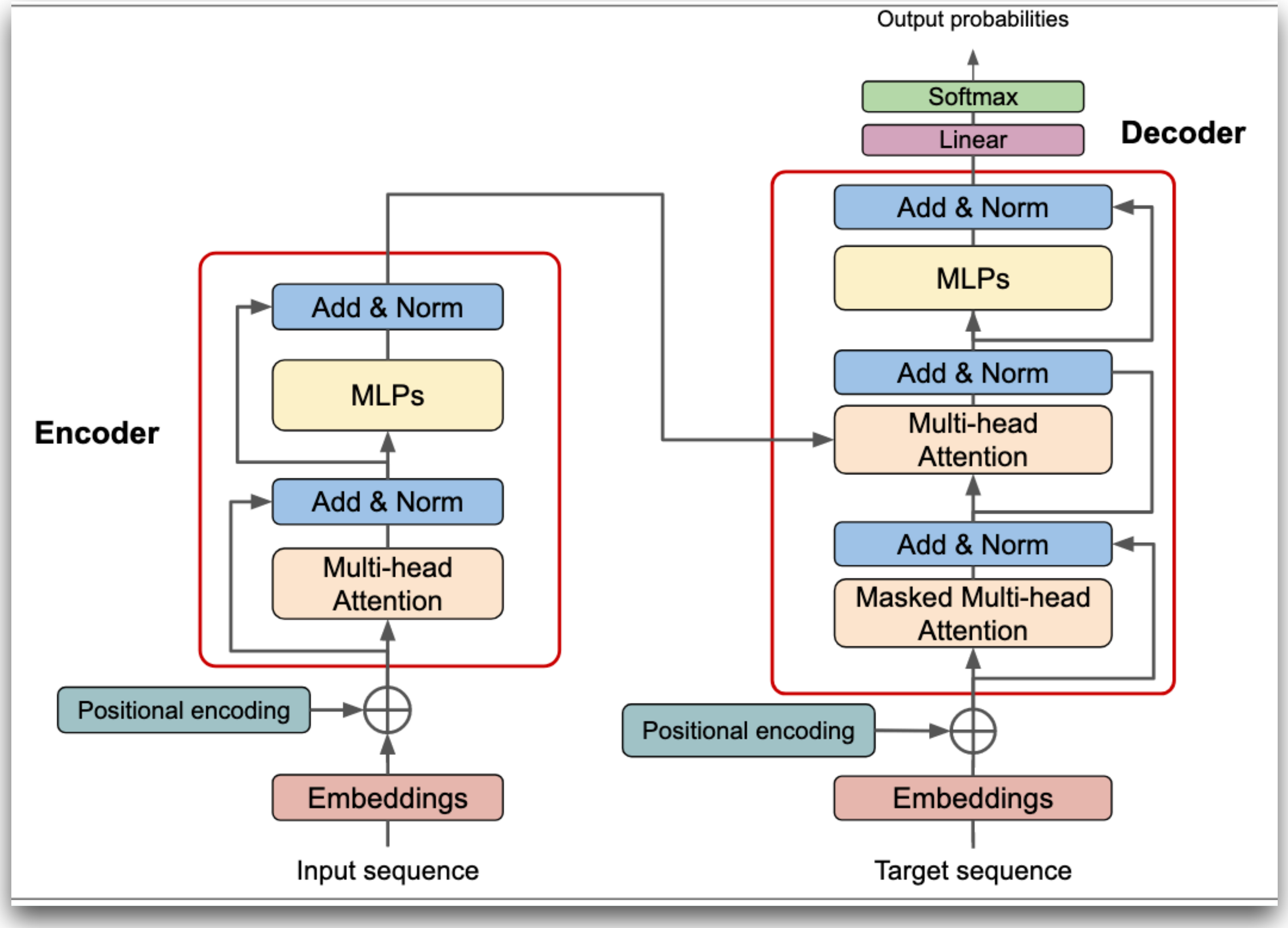
Handling long-distance dependencies through managing memory

- **Motivation:** Vanilla RNNs struggled with *long-term dependencies* because of vanishing/exploding gradients during training.
- **Usefulness:** LSTMs use *gated cells* to *selectively remember or forget information*, allowing **stable learning over long sequences**—powering early breakthroughs in speech recognition, translation, and text generation.



Transformers

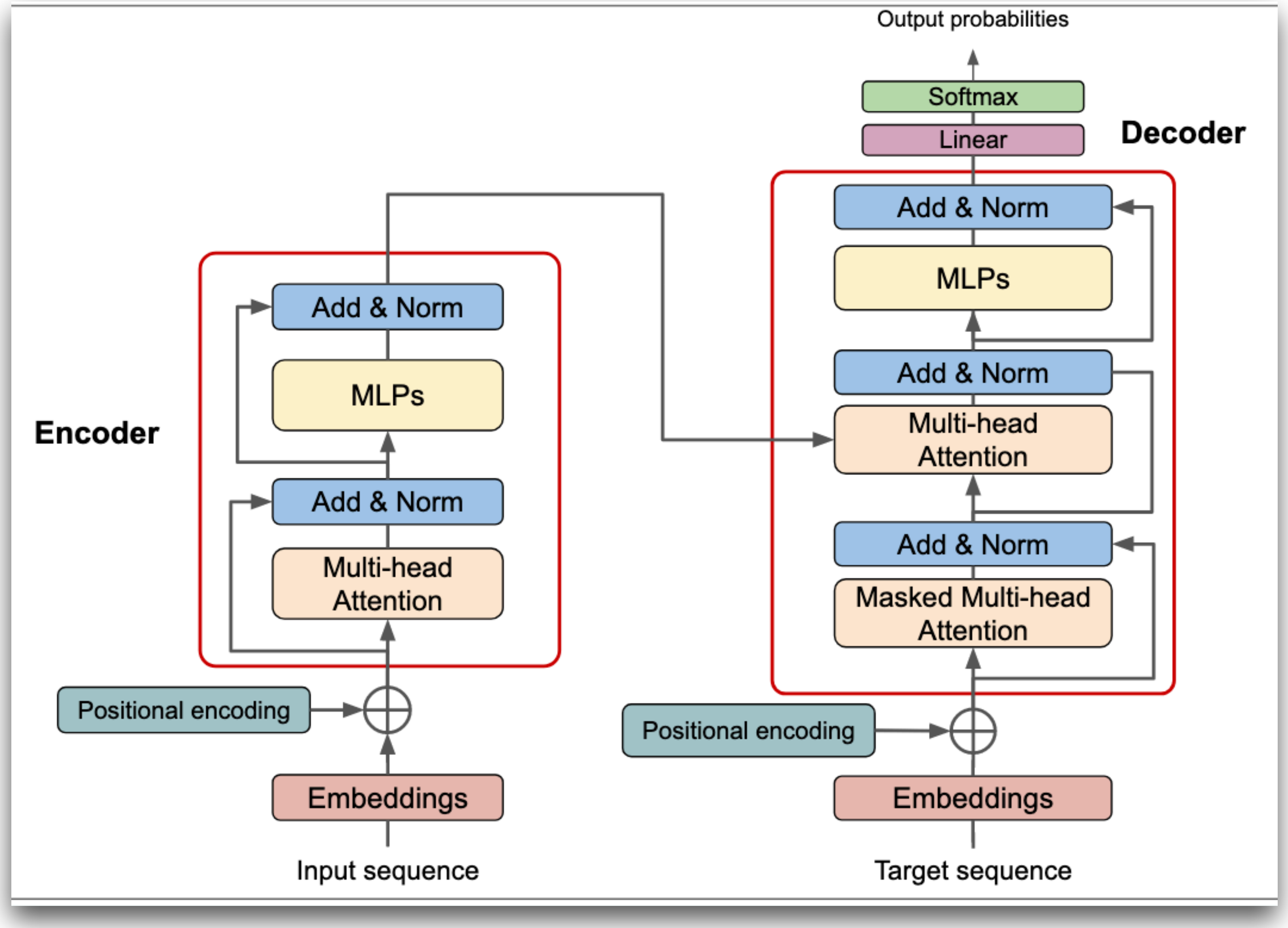
Global information access



Transformers

Global information access

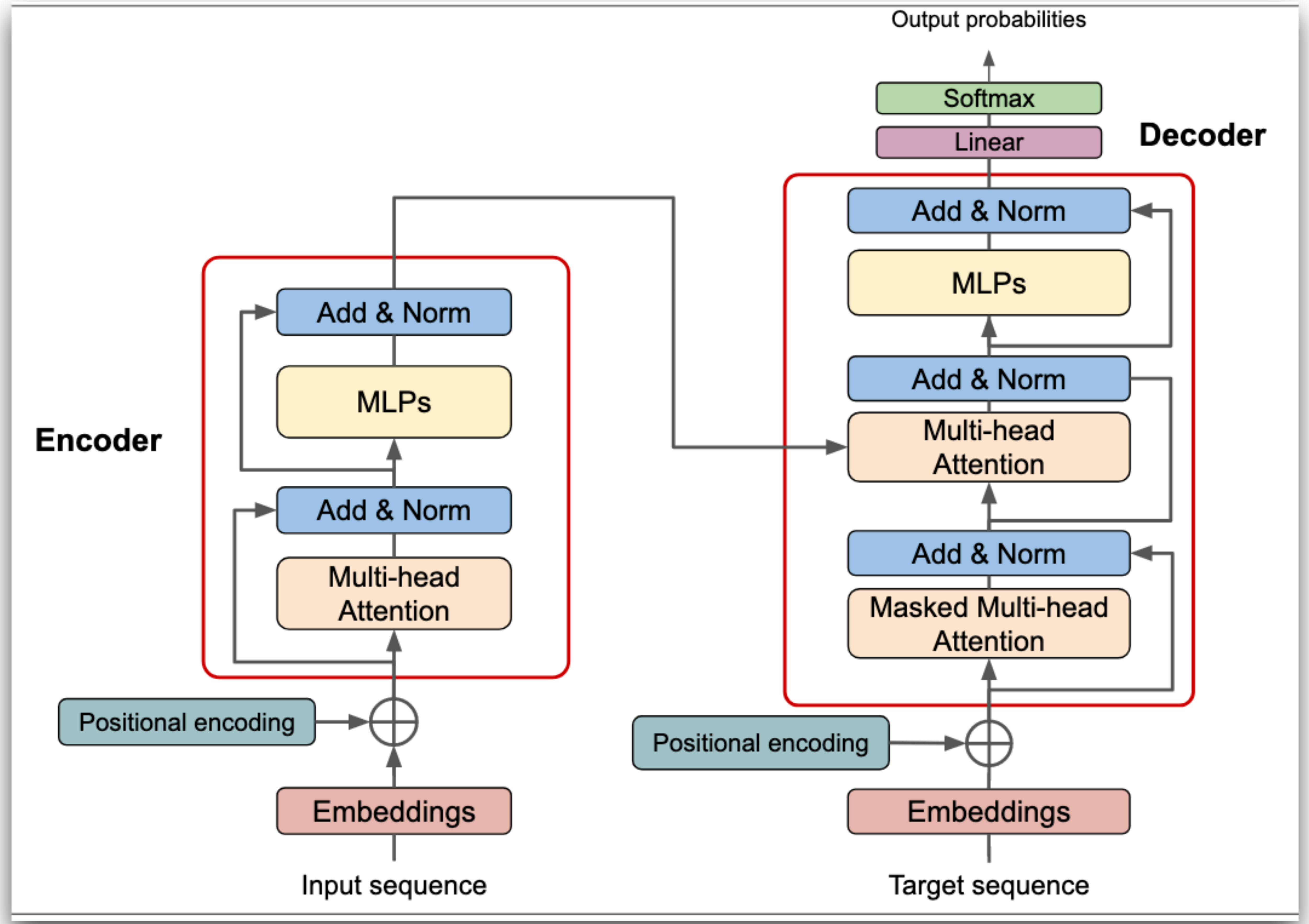
- **Motivation:** Even LSTMs process sequences step-by-step, limiting *parallelization* and *global context* access.



Transformers

Global information access

- **Motivation:** Even LSTMs process sequences step-by-step, limiting *parallelization* and *global context* access.
- **Usefulness:** Transformers replace recurrence with ***self-attention***, letting the model **directly relate every token to every other**—making large-scale training efficient and forming the foundation of modern large language models.



Next Time (after fall break): Tom McCoy presenting on LLMs

Thank you for listening!

Slides are partially adapted from Bob Frank (left) and Tom McCoy (right)

And their slides are adpated from Jurafsky and Martin: https://web.stanford.edu/~jurafsky/slp3/